Chaos in a Pendulum

Simulation,
Experimental Characterization,
and Prediction of
Deterministic Periodic and Chaotic States

27th March 2002

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Galileo's Apocryphal Chandelier

$$T = 2\pi\sqrt{\frac{L}{g}}$$

The Plane Pendulum

Newton's Second Law:

(Torque about support axis) =
(Moment of inertia) * (Angular acceleration)

$$\text{Torque} = L\,F = I\left(\frac{d^2\theta}{dt^2}\right) = (m\,L^2)\left(\frac{d^2\theta}{dt^2}\right)$$

Important -- The restoring force is nonlinear:

$$F = -m\,g\sin\theta$$

The Damped and Driven Pendulum

Swinging under the influence of:

- gravity (restoring force, "-")
- velocity-proportional damping (resisting force, "-")
- sinusoidal driver ("+")

The equation of motion (relating the above torques):

$$(mL^2)\left(\frac{d^2\theta}{dt^2}\right) = -m\,g\,L\sin\theta - b\left(\frac{d\theta}{dt}\right) + T\sin(\omega_f\,t)$$

After some intense algebra:

$$\left(\frac{d^2\theta}{dt^2}\right) = -\left(\frac{g}{L}\right)\sin\theta - \left(\frac{b}{mL^2}\right)\left(\frac{d\theta}{dt}\right) + \left(\frac{T}{mL^2}\right)\sin(\omega_f\,t)$$

The Damped and Driven Pendulum

(continued)

Natural units for the pendulum:

$$t' = \omega_0 t \implies dt' = \omega_0 dt$$

$$\omega_d \equiv \omega_f / \omega_0$$

$$\omega_0^2 \equiv g / L$$

Then fiddling with derivatives and constants turns this

$$\left(\frac{d^2\theta}{dt^2}\right) = -\left(\frac{g}{L}\right)\sin\theta - \left(\frac{b}{mL^2}\right)\left(\frac{d\theta}{dt}\right) + \left(\frac{T}{mL^2}\right)\sin(\omega_f t)$$

into that

$$\left(\frac{d^2\theta}{dt'^2}\right) = -\sin\theta - \left(\frac{1}{q}\right)\left(\frac{d\theta}{dt'}\right) + (g)\sin(\omega_d t')$$

where

$$\frac{1}{q} = \left(\frac{1}{\omega_0}\right)\left(\frac{b}{I}\right) \qquad g = \frac{T}{\omega_0^2 I} = \frac{T}{T_{critical}} \qquad \omega_d = \frac{\omega_f}{\omega_0}$$

The Real (Physical) Pendulum

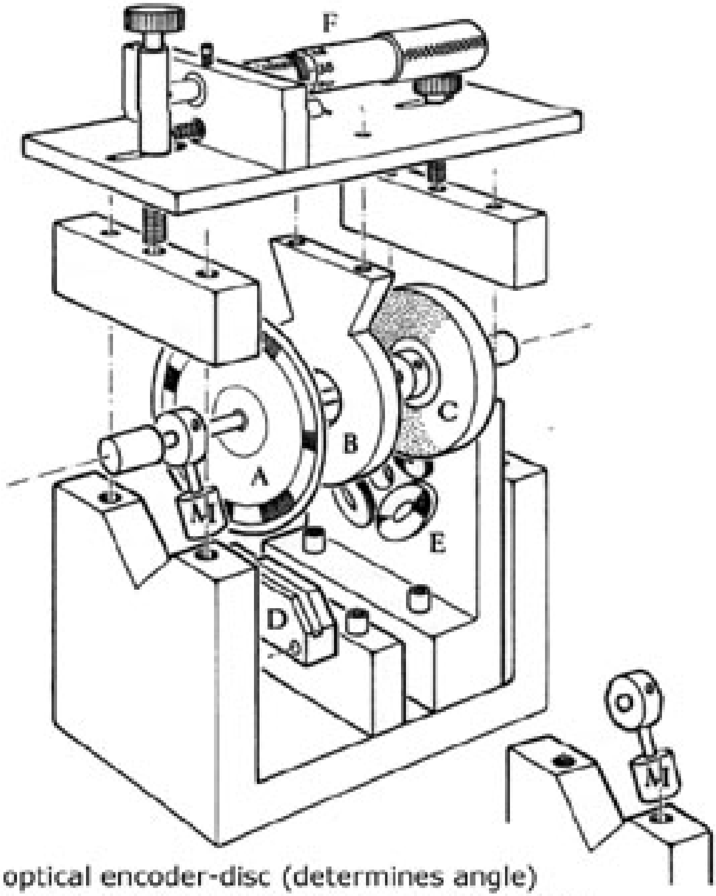
Getting physical:

- pendulum: length = 2 cm; eddy-current damping (controlled by a micrometer); optical encoding of angular position; brushless slotless linear AC/DC motor

- driver: sinusoidal torque; frequency = 0.3 - 3.0 Hz, +/- 0.001 Hz; output voltage = 0 - 7 V

- PC card and software

"Exploded" View of the Chaotic Pendulum



A - optical encoder-disc (determines angle)
B - stationary copper plate (used for damping)
C - ring magnet (used for damping)
D - light emitter-detector assembly (reads angle)
E - two pairs of drive coils (parts of AC/DC motor)
F - micrometer (used to control damping)
M - pendulum mass (+ short rod = our "pendulum")

The Physical Pendulum

(continued)

Seen that already:

$$\left(\frac{d^2\theta}{dt'^2}\right) = -\sin\theta - \left(\frac{1}{q}\right)\left(\frac{d\theta}{dt'}\right) + (g)\sin(\omega_d t')$$

The good old dimensionless parameters:

$$\frac{1}{q} = \left(\frac{1}{\omega_0}\right)\left(\frac{b}{I}\right) \qquad g = \frac{T}{\omega_0^2 I} = \frac{T}{T_{critical}} \qquad \omega_d = \frac{\omega_f}{\omega_0}$$

But what about ω_0, and b, and I, and T, and $T_{critical}$?

Parameterization of the Pendulum

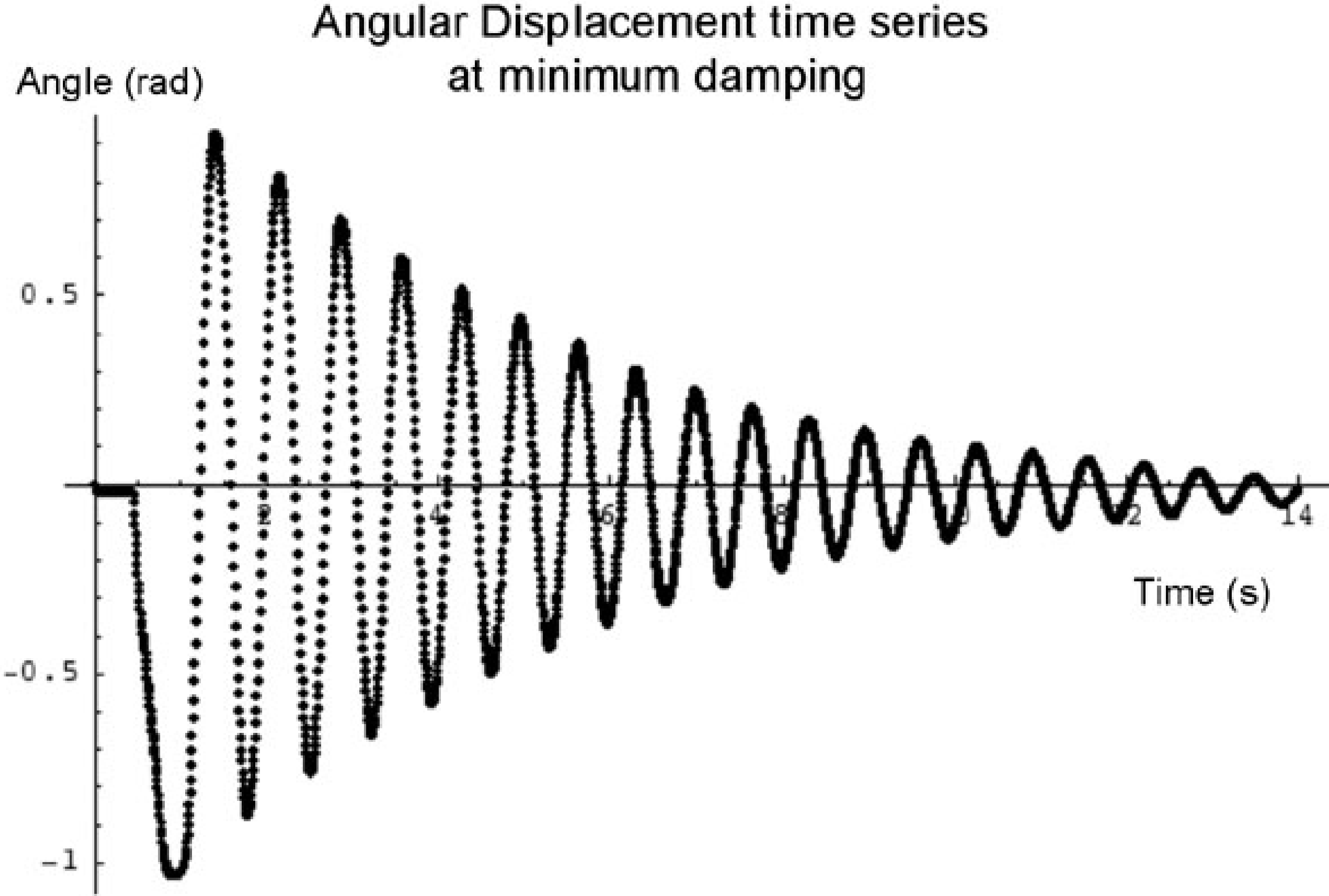
Determining the Natural Frequency, ω_0:

- collecting time series of angular displacement, θ, at minimum damping

- performing nonlinear fitting of damping equation to part of the time series

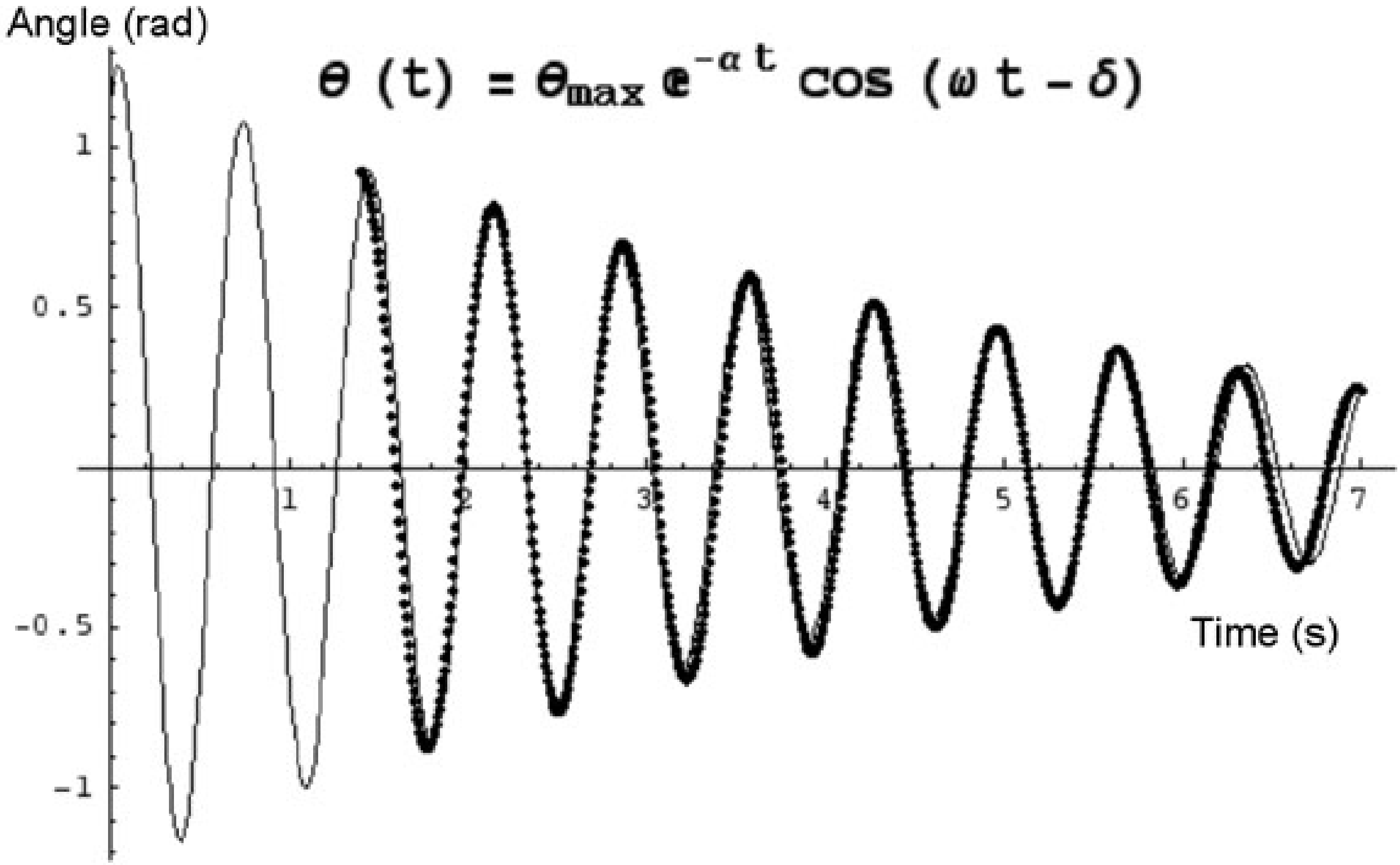
- calculating ω_0 from the obtained ω and α, damping parameter: $\omega_0 = \sqrt{\omega^2 + \alpha^2}$

Now the driving frequency parameter can be calculated and varied ➔ $\omega_d = \frac{\omega_f}{\omega_0}$



Angular Displacement time series
at minimum damping
Angle (rad)
0.5
-0.5
-1
2
4
6
8
14
Time (s)

Data and nonlinear fit for minimum damping



Parameterization of the Pendulum

(continued)

b/I as a function of Micrometer Setting, S:

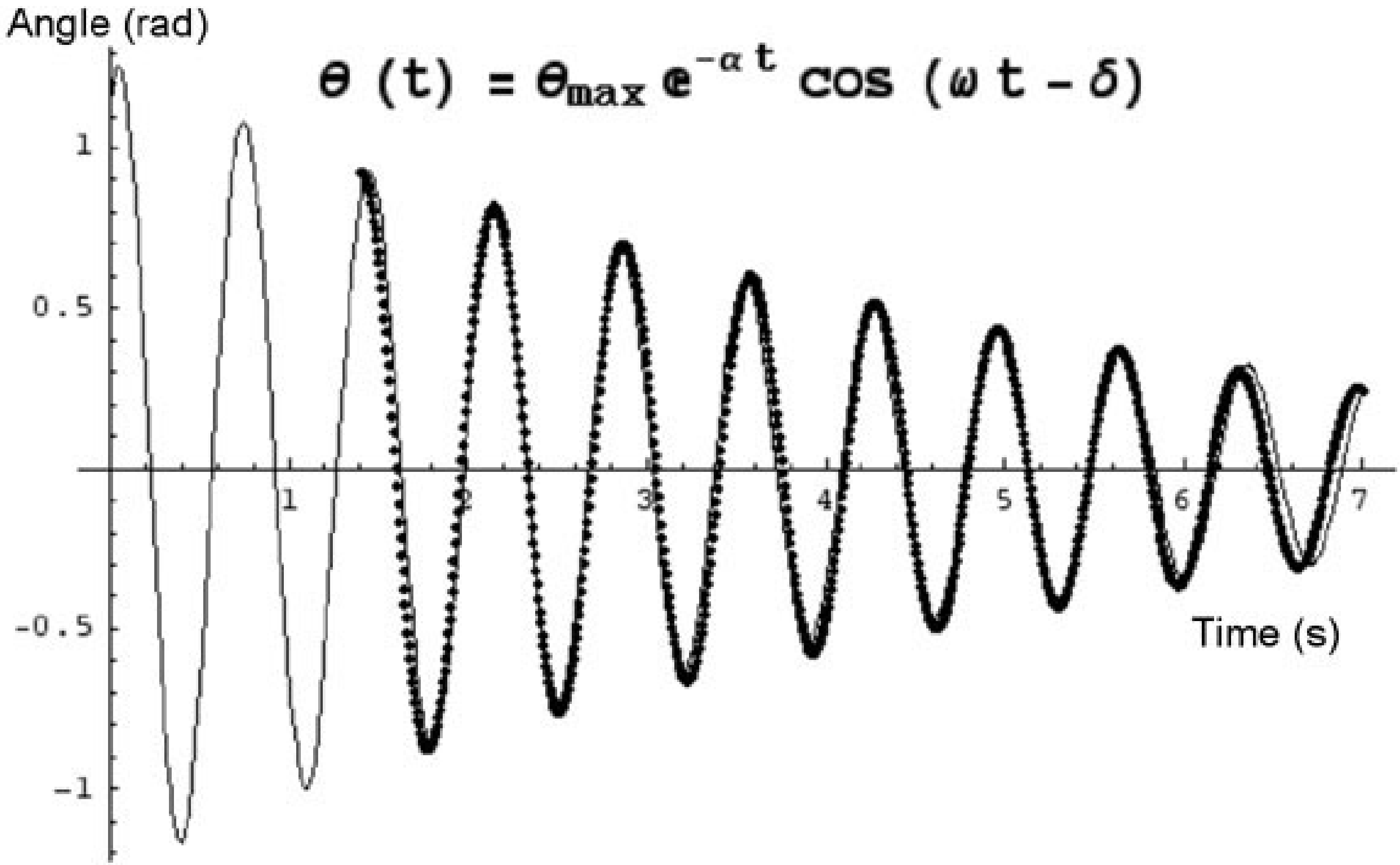
- collecting time series of angular displacement, θ, at various values of S
- performing nonlinear fitting of damping equation to part of the time series for each S

$$(b/I) = 2\alpha$$

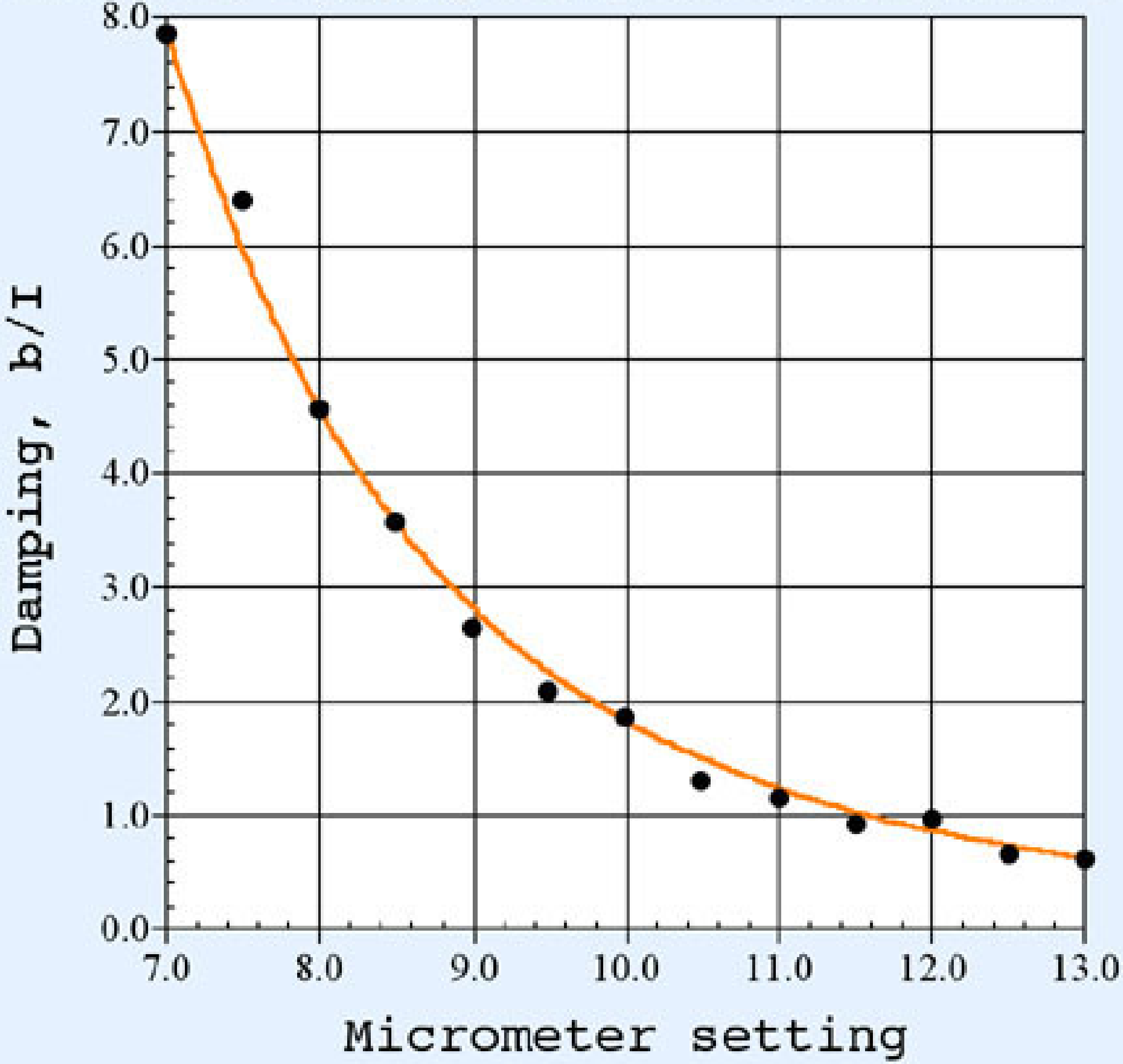
- obtaining calibration curve for various micrometer settings

Now the damping parameter can be calculated and varied ➔ $\frac{1}{q} = \left(\frac{1}{\omega_0}\right)\left(\frac{b}{I}\right)$

Data and nonlinear fit for minimum damping



Normalized Damping as a function of Micrometer Setting



Parameterization of the Pendulum

(continued)

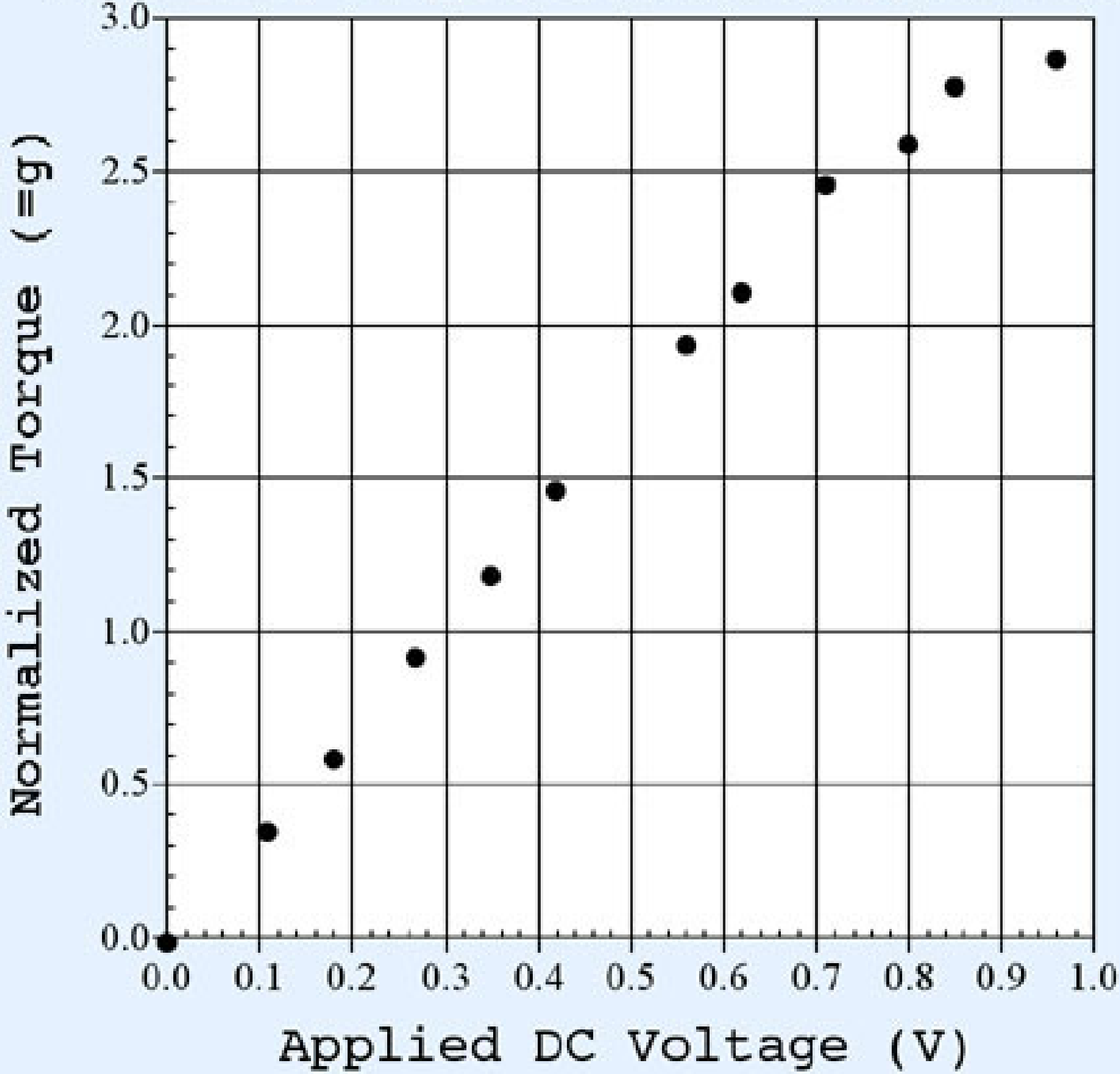
Normalized Torque as a function of Driving Voltage (DC):

- pendulum turned "on-edge" to eliminate gravitational restoring force
- measuring terminal angular velocity, ω_{term} , at constant damping and varying drive voltage
- obtaining calibration curve for normalized torque, $T/T_{critical}$, as a function of voltage (DC)

Now the driving amplitude parameter can be calculated and varied ➔

$$g = \frac{T}{\omega_0{}^2 I} = \frac{T}{T_{critical}}$$

Normalized Torque as a function of Applied Voltage



Tweaking the Parameters

Driving frequency
(kept constant,
$\omega_d = 2/3$)

$$\omega_d = \frac{\omega_f}{\omega_0}$$

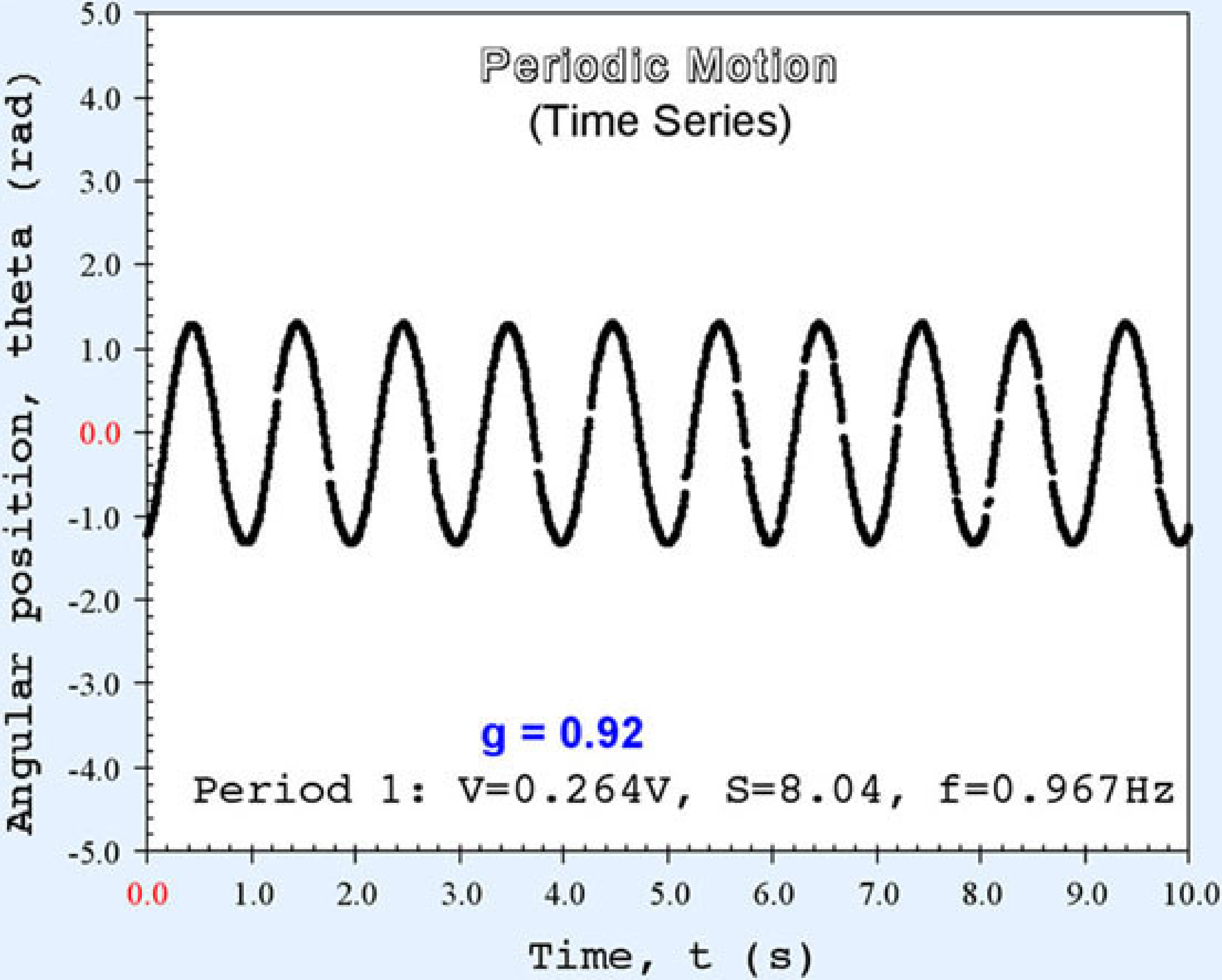
Micrometer damping
(two different settings,
q = 2 and q = 4)

$$\frac{1}{q} = \left(\frac{1}{\omega_0}\right)\left(\frac{b}{I}\right)$$

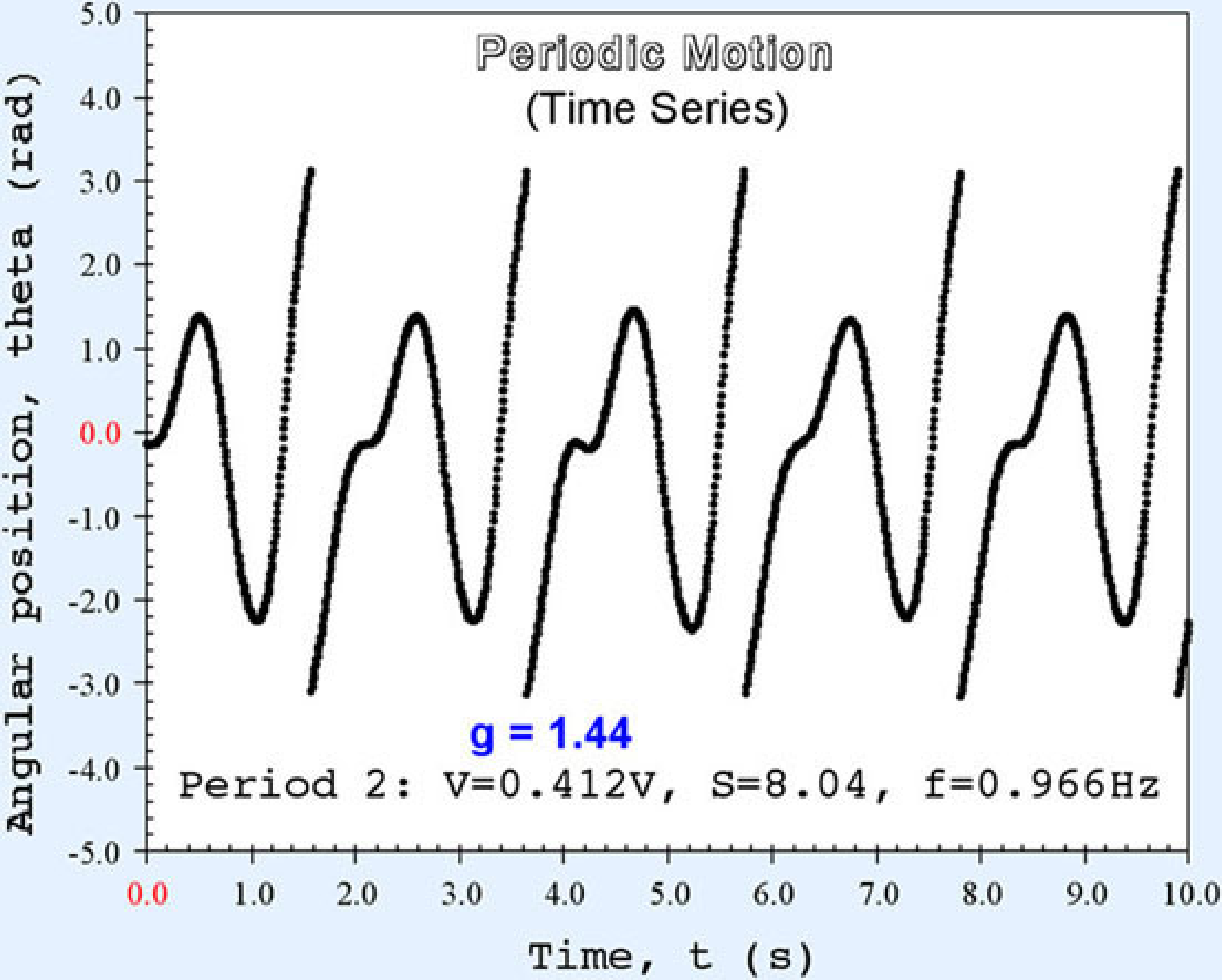
Driving amplitude
(a range of settings)

$$g = \frac{T}{\omega_0^2 I} = \frac{T}{T_{critical}}$$

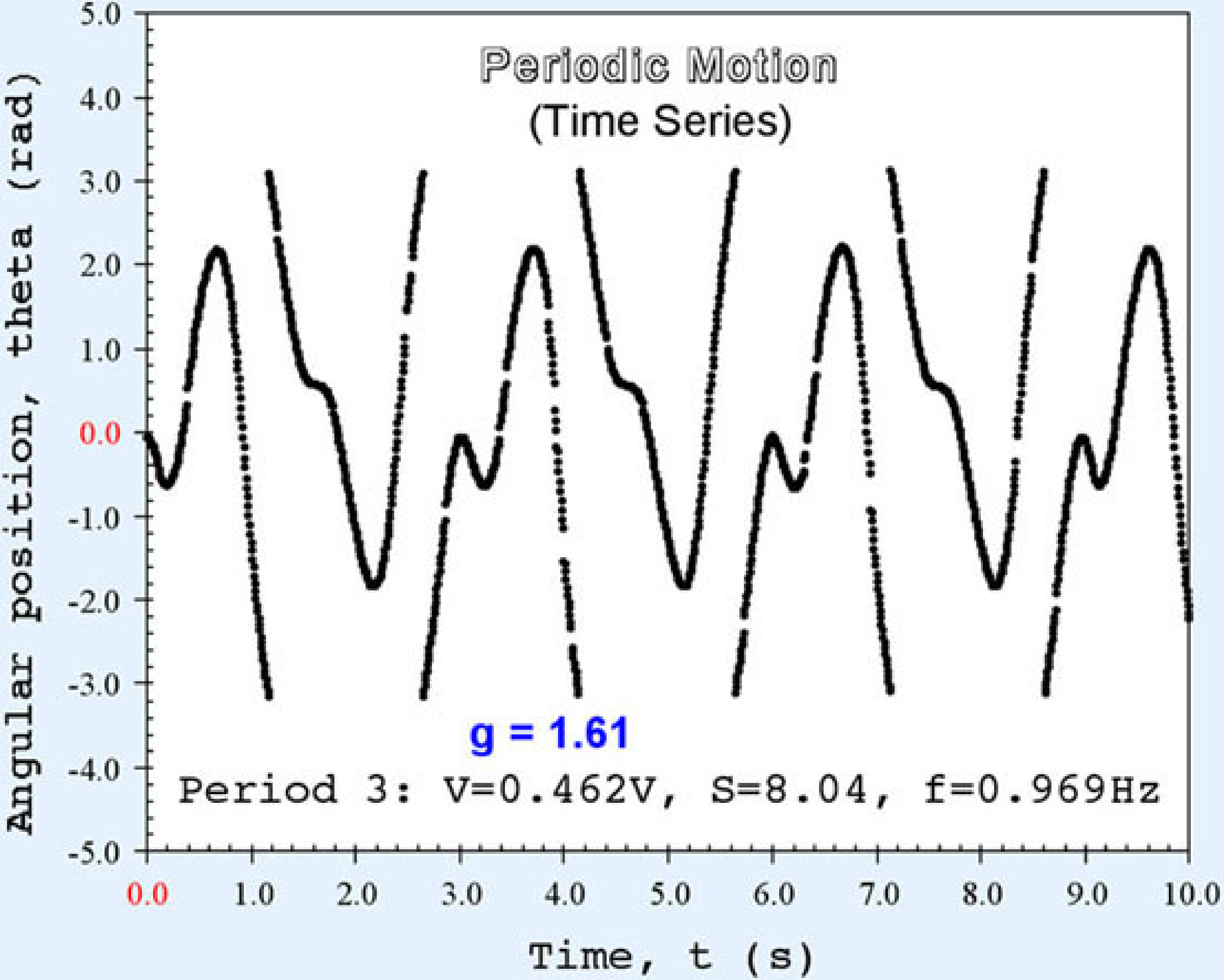
g = 0.92	g = 1.44	g = 1.61	g = 1.75
Period 1	**Period 2**	**Period 3**	**Chaos**



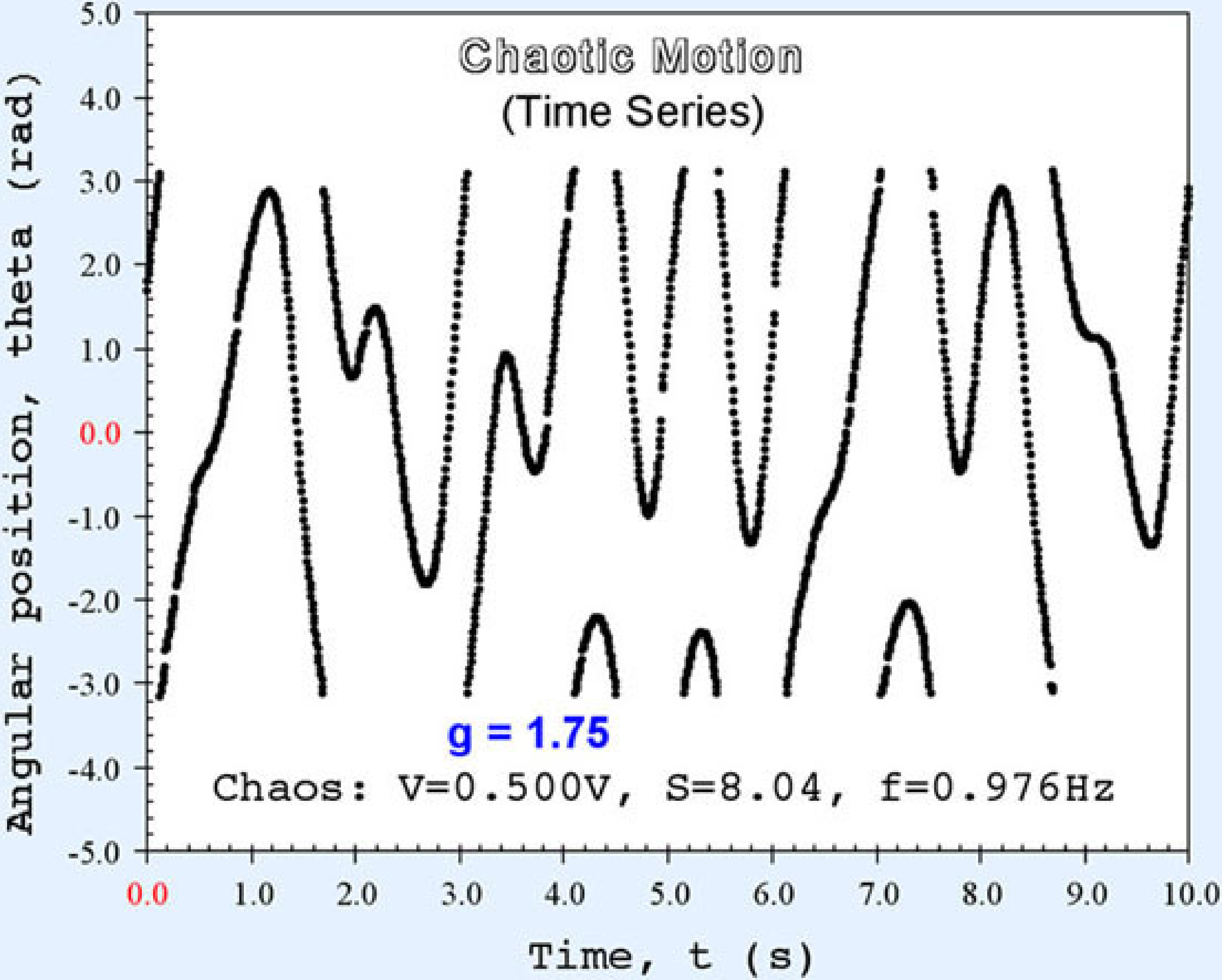
Periodic Motion
(Time Series)
Angular position, theta (rad)
5.0
4.0
3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
-4.0
-5.0
g = 0.92
Period 1: V=0.264V, S=8.04, f=0.967Hz
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
Time, t (s)



Periodic Motion
(Time Series)
g = 1.44
Period 2: V=0.412V, S=8.04, f=0.966Hz
Angular position, theta (rad)
Time, t (s)
5.0
4.0
3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
-4.0
-5.0
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0



Periodic Motion
(Time Series)
g = 1.61
Period 3: V=0.462V, S=8.04, f=0.969Hz
Angular position, theta (rad)
Time, t (s)
5.0
4.0
3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
-4.0
-5.0
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0



Chaotic Motion
(Time Series)
g = 1.75
Chaos: V=0.500V, S=8.04, f=0.976Hz
Angular position, theta (rad)
Time, t (s)
5.0
4.0
3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
-4.0
-5.0
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0

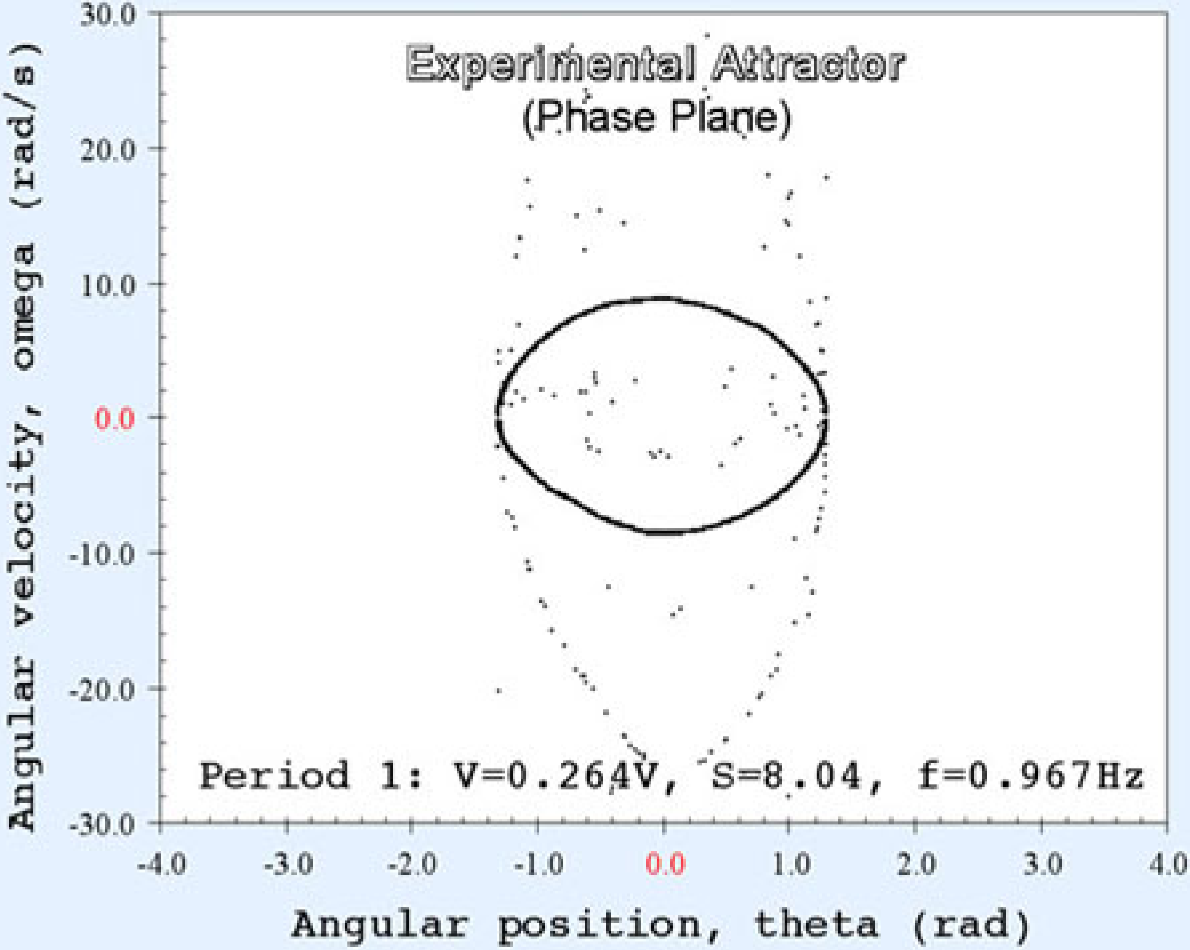
CHAOS

Aperiodic long-term behavior in a

deterministic system that exhibits

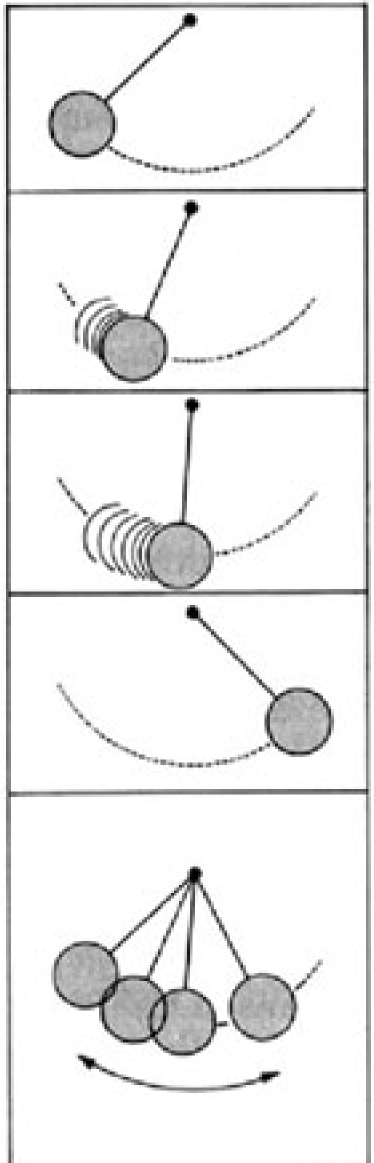
sensitivity to initial conditions.

Phase Portrait



How do we read the Phase Portrait ?

How to read the Phase Plane diagram

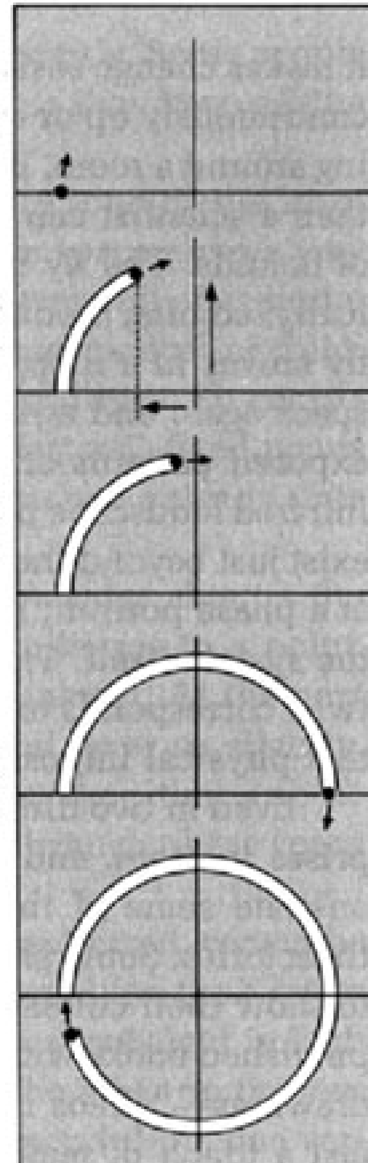


Velocity is zero as the pendulum starts its swing. Position is a negative number, the distance to the left of the center.

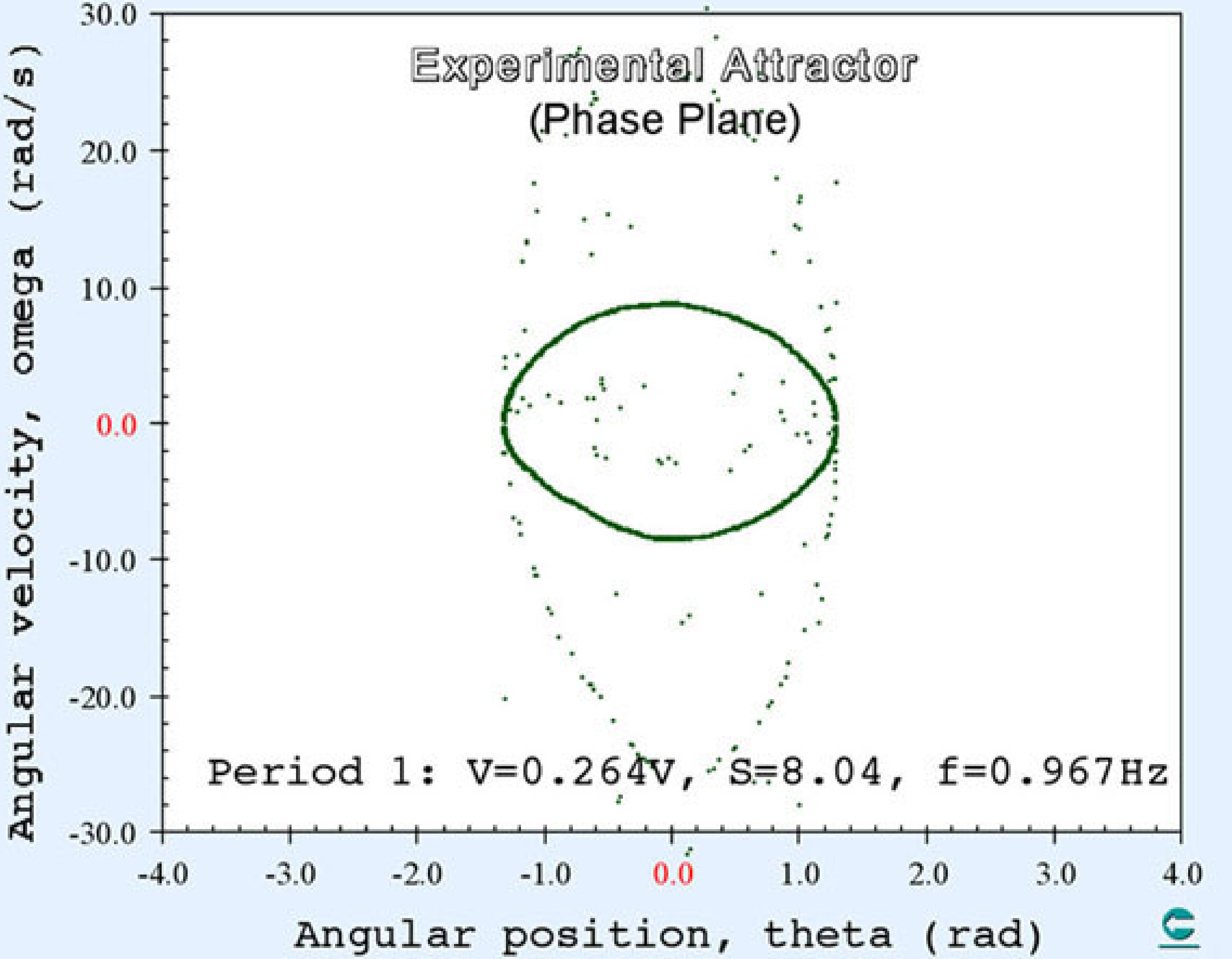
The two numbers specify a single point in two-dimensional phase space.

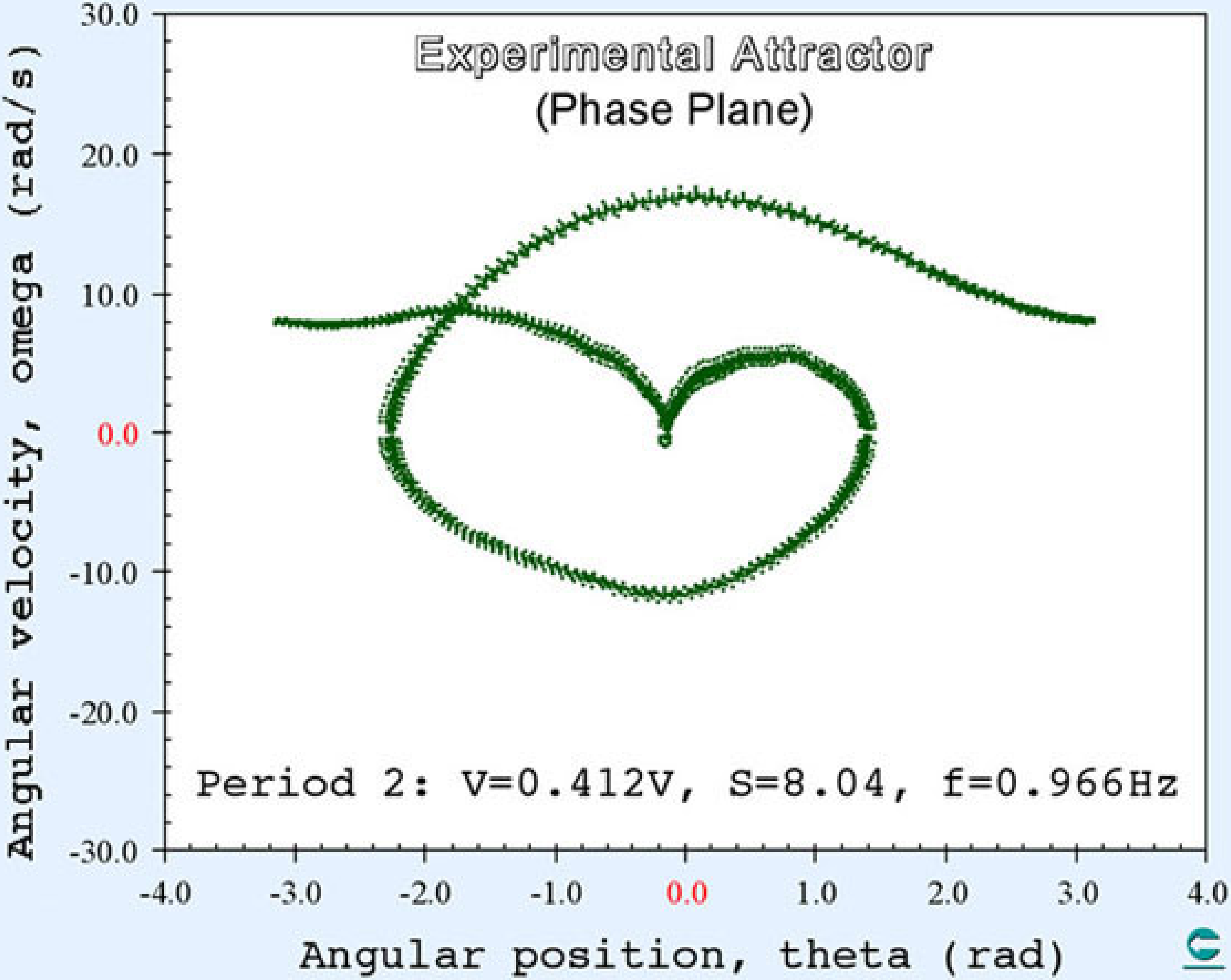
Velocity reaches its maximum as the pendulum's position passes through zero.

Velocity declines again to zero, and then becomes negative to represent leftward motion.

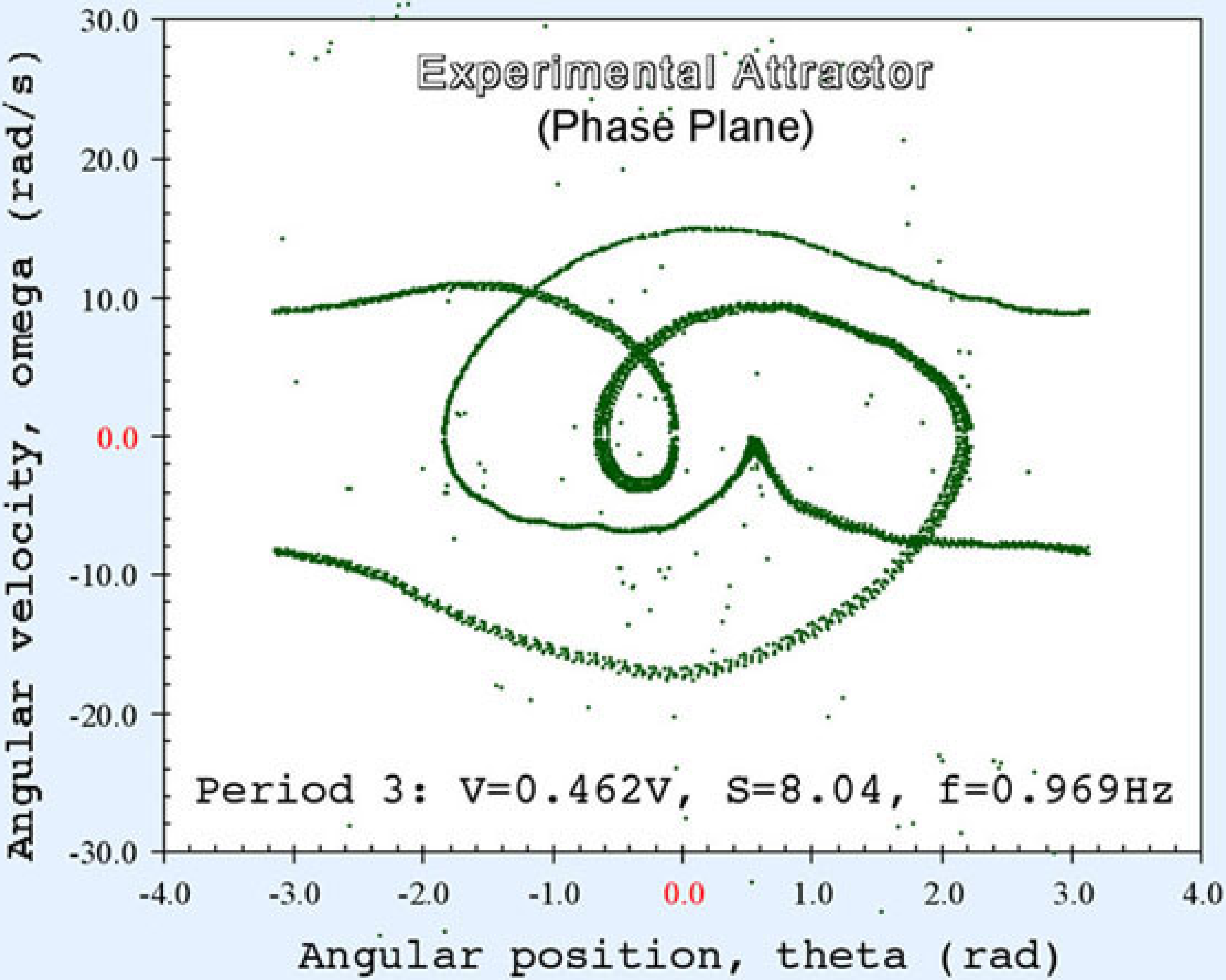


Adolph E. Brotman


Experimental Attractor
(Phase Plane)
Angular velocity, omega (rad/s)
30.0
20.0
10.0
0.0
-10.0
-20.0
-30.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0
Angular position, theta (rad)
Period 1: V=0.264V, S=8.04, f=0.967Hz



Experimental Attractor
(Phase Plane)
Period 2: V=0.412V, S=8.04, f=0.966Hz
Angular velocity, omega (rad/s)
Angular position, theta (rad)
30.0
20.0
10.0
0.0
-10.0
-20.0
-30.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0


Experimental Attractor
(Phase Plane)
Period 3: V=0.462V, S=8.04, f=0.969Hz
Angular velocity, omega (rad/s)
Angular position, theta (rad)
30.0
20.0
10.0
0.0
-10.0
-20.0
-30.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0

Pendulum Bifurcation Diagrams

Solving the differential equation of motion numerically:

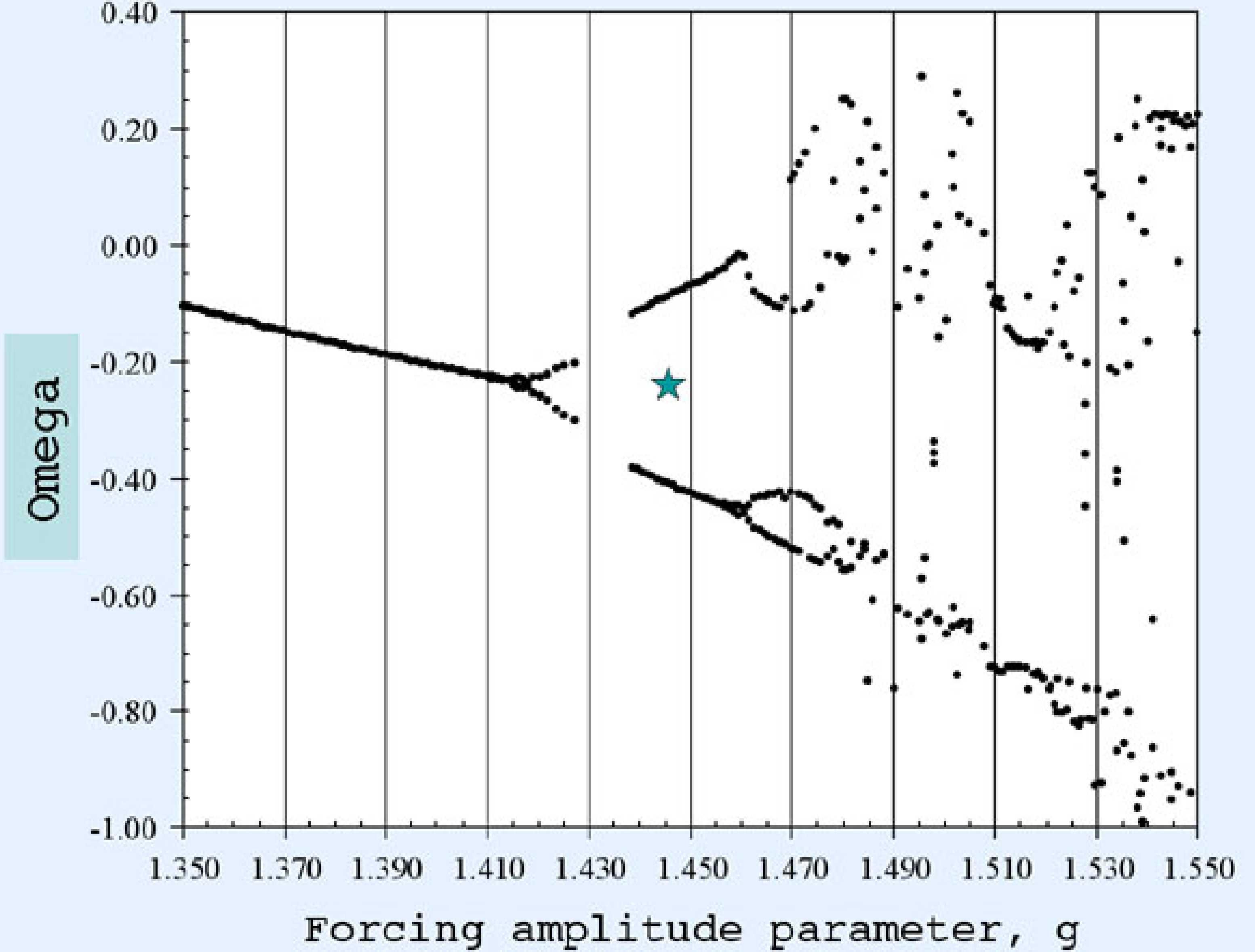
$$\left(\frac{d^2\theta}{dt'^2}\right) = -\sin\theta - \left(\frac{1}{q}\right)\left(\frac{d\theta}{dt'}\right) + (g)\sin(\omega_d t')$$

becomes

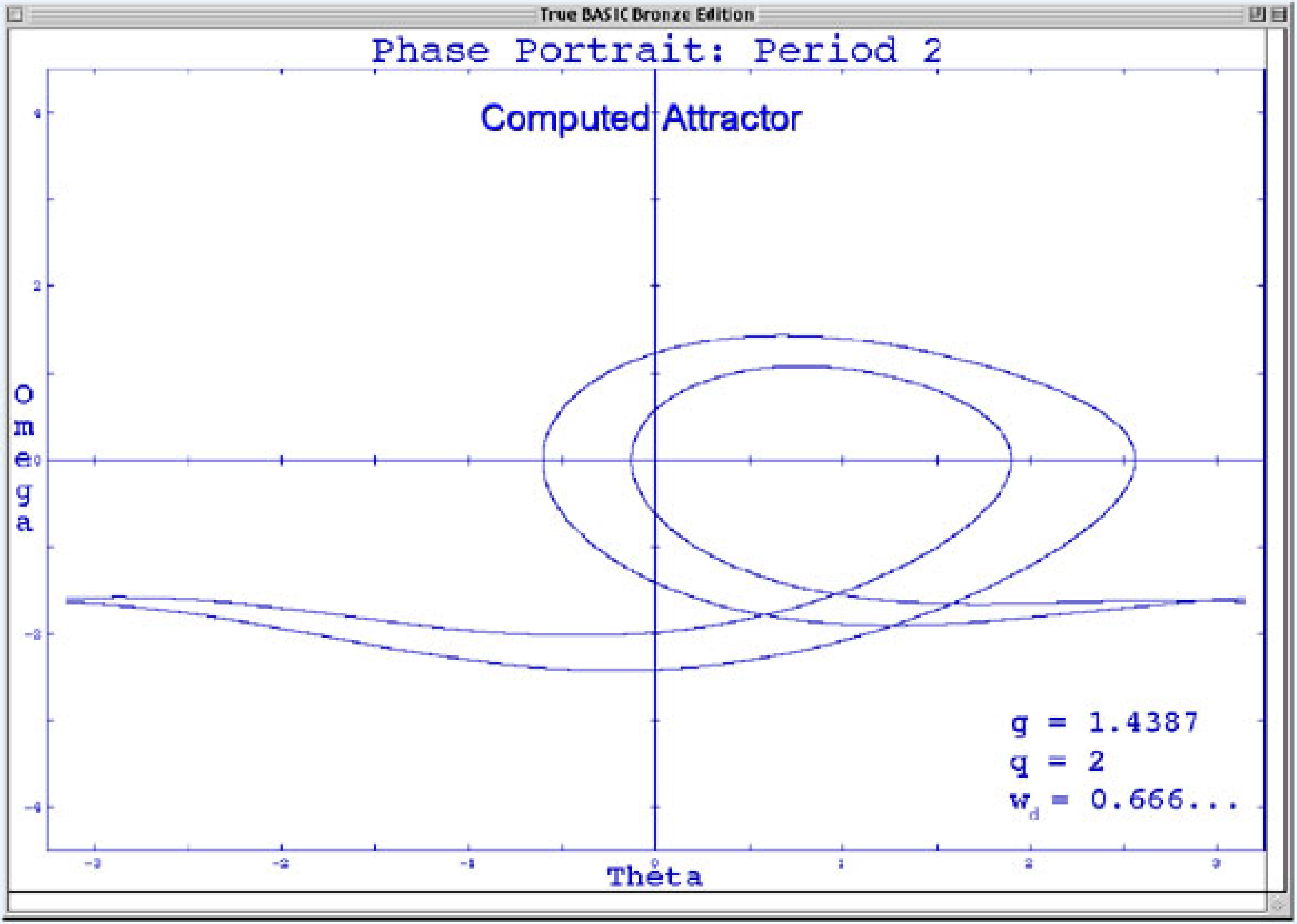
$$accel = -\sin x - \left(\frac{1}{q}\right) v + g \sin(w t')$$

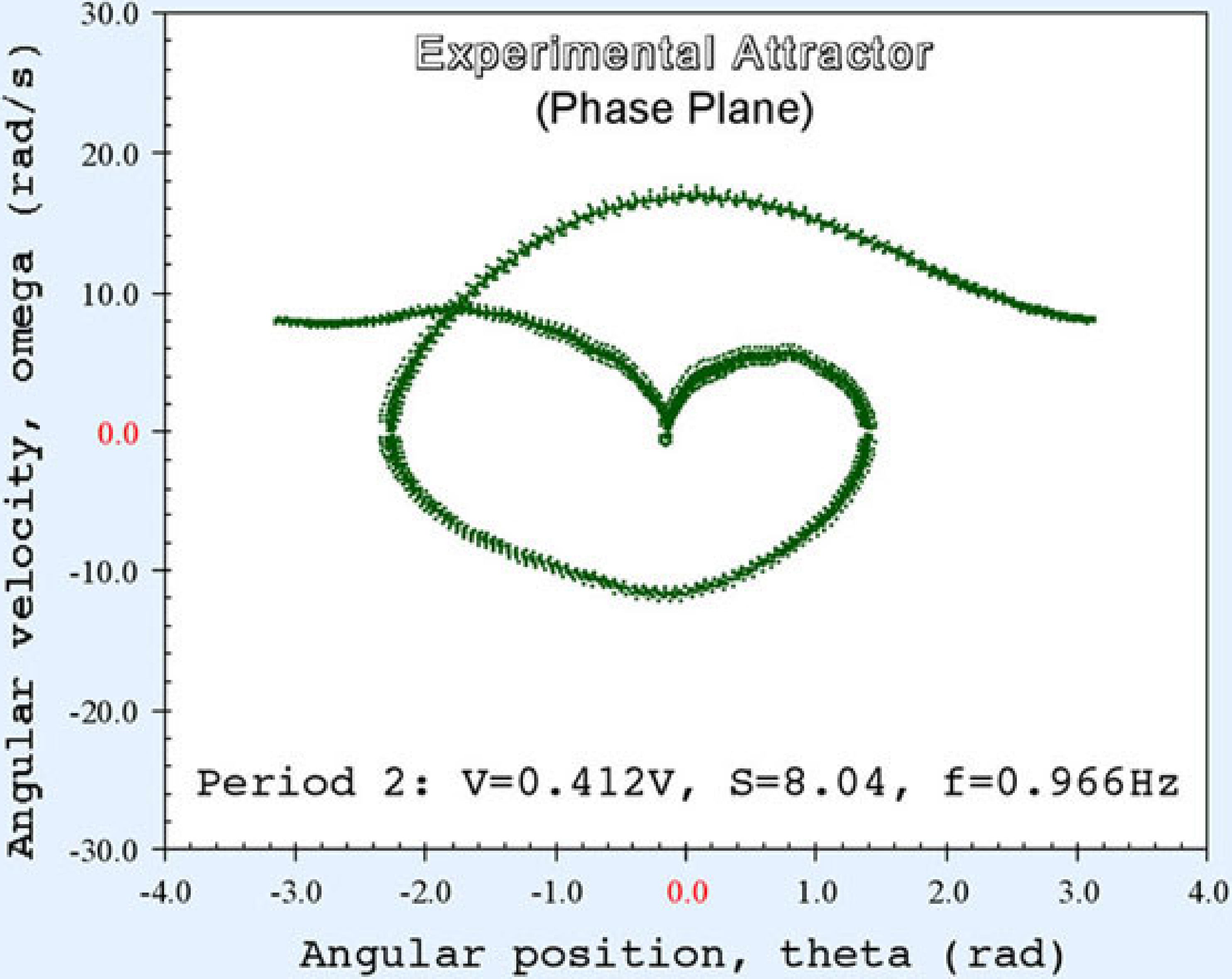
I have used the fourth-order *Runge-Kutta* method, with an adjustable time increment.

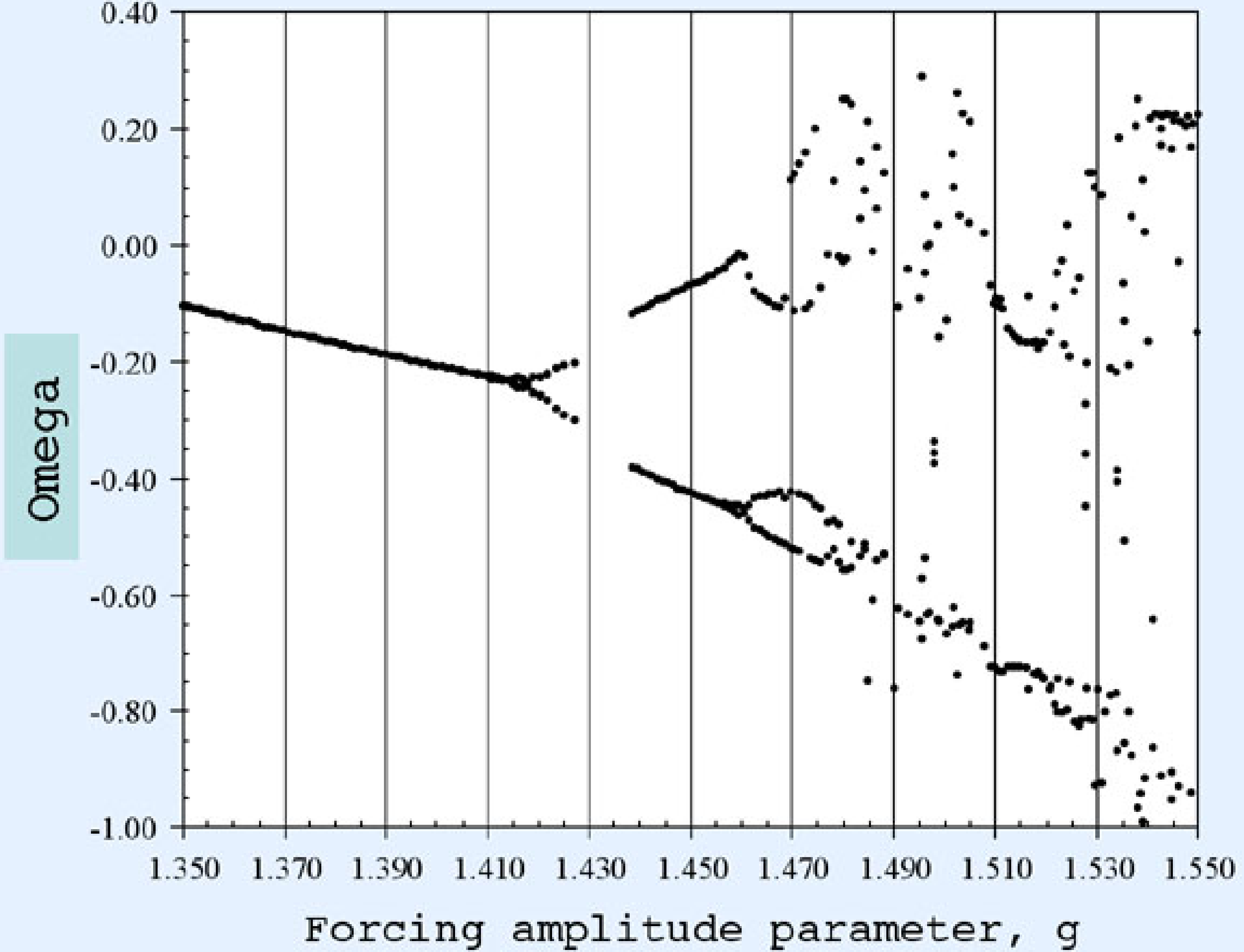
Solving numerically for x and for v, at small increments of g.



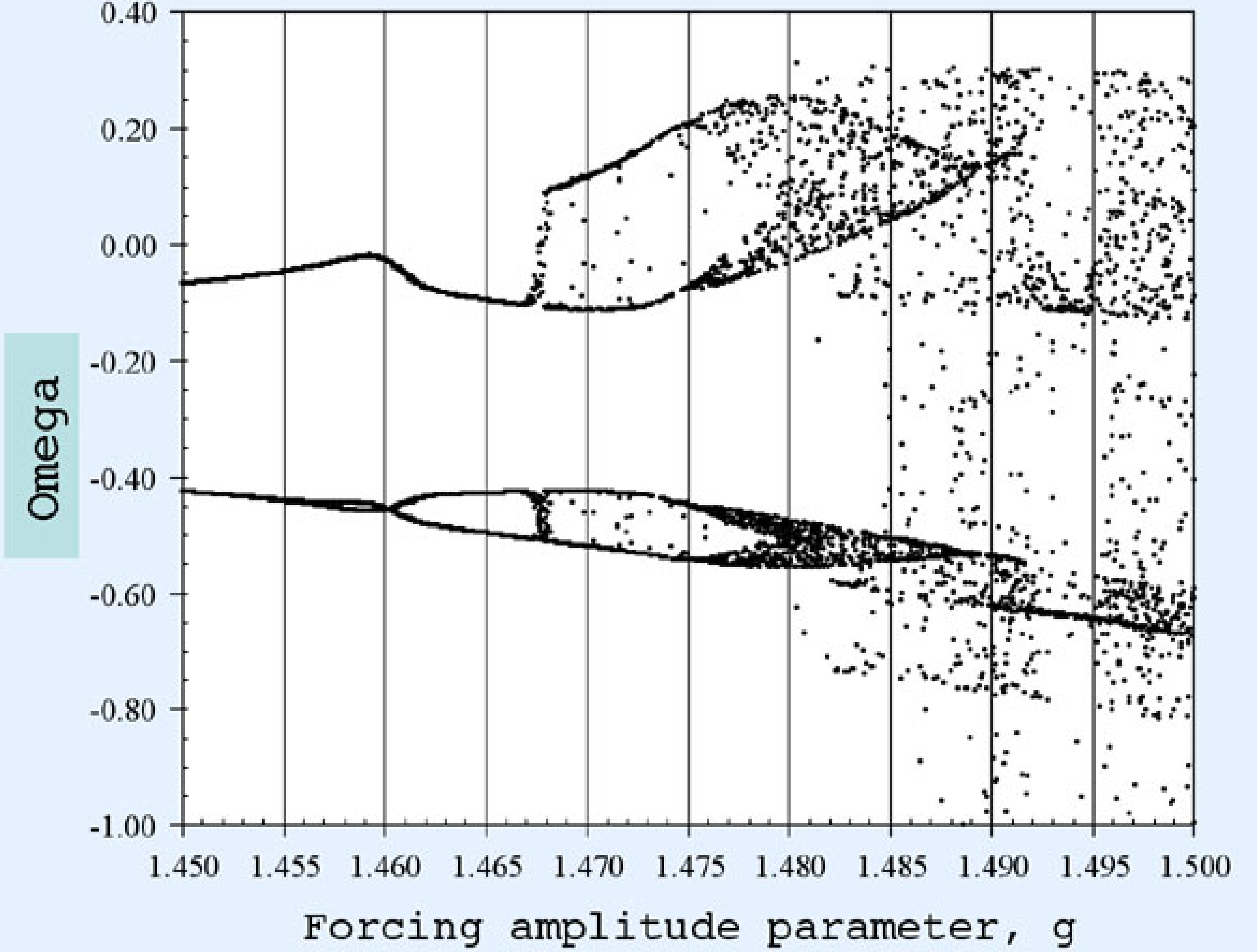
0.40
0.20
0.00
-0.20
-0.40
-0.60
-0.80
-1.00
Omega
1.350
1.370
1.390
1.410
1.430
1.450
1.470
1.490
1.510
1.530
1.550
Forcing amplitude parameter, g


True BASIC Bronze Edition
Phase Portrait: Period 2
Computed Attractor
Omega
Theta
q = 1.4387
q = 2
w_d = 0.666...


Experimental Attractor
(Phase Plane)
Period 2: V=0.412V, S=8.04, f=0.966Hz
Angular velocity, omega (rad/s)
Angular position, theta (rad)
30.0
20.0
10.0
0.0
-10.0
-20.0
-30.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0



0.40
0.20
0.00
-0.20
-0.40
-0.60
-0.80
-1.00
Omega
1.350
1.370
1.390
1.410
1.430
1.450
1.470
1.490
1.510
1.530
1.550
Forcing amplitude parameter, g



0.40
0.20
0.00
-0.20
-0.40
-0.60
-0.80
-1.00
Omega
1.450
1.455
1.460
1.465
1.470
1.475
1.480
1.485
1.490
1.495
1.500
Forcing amplitude parameter, g

Prerequisites for Chaotic Behavior in a Physical System

1. at least three independent dynamical variables

2. nonlinearity in the equations of motion

3. appropriate values for the parameters

The Chaotic Pendulum (again...)

$$\left(\frac{d^2\theta}{dt'^2}\right) = -\sin\theta - \left(\frac{1}{q}\right)\left(\frac{d\theta}{dt'}\right) + (g)\sin(\omega_d t')$$

1. at least three independent dynamical variables

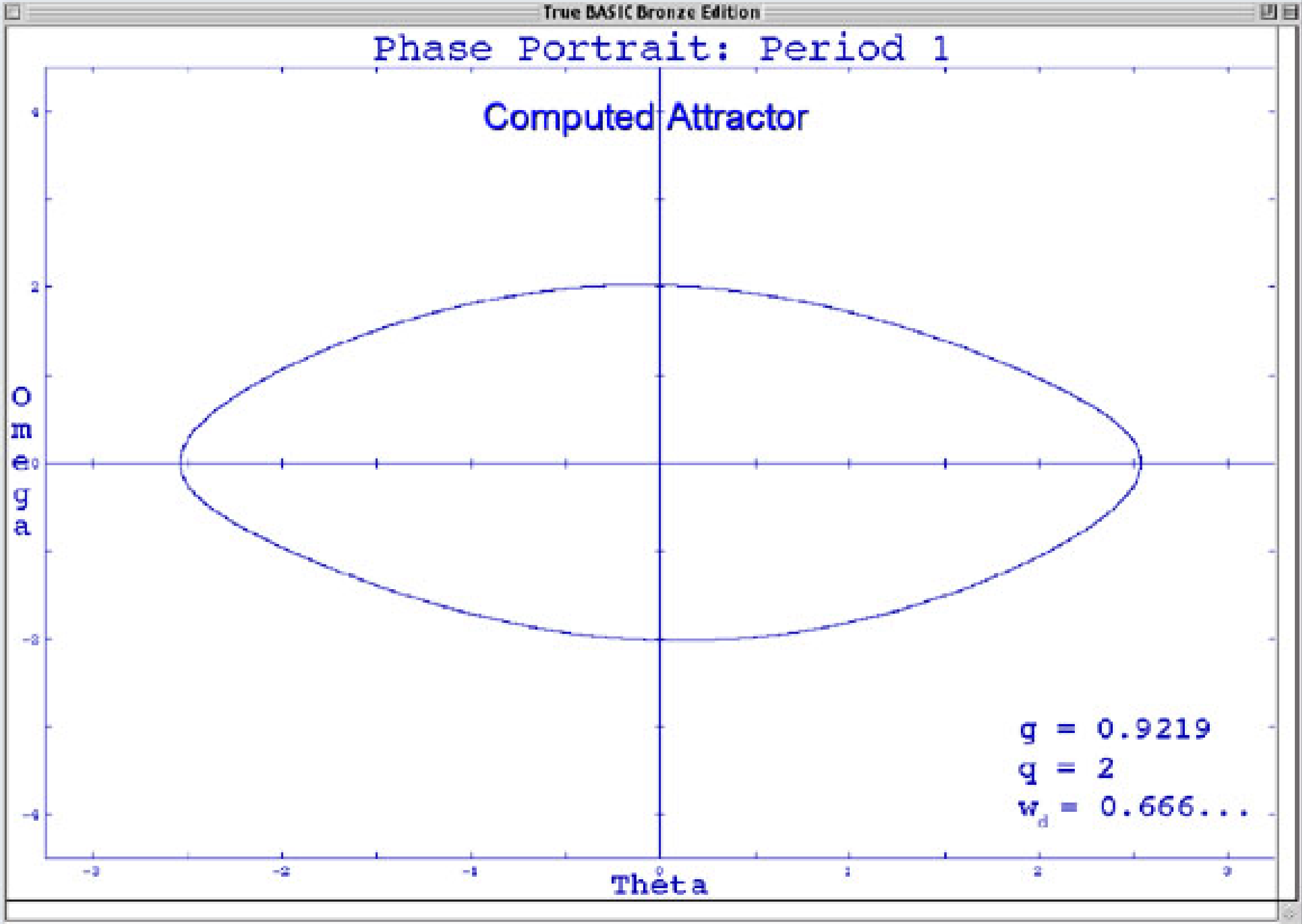
$$\frac{d\omega}{dt} = -\frac{\omega}{q} - \sin\theta + g\sin\varphi \qquad \frac{d\theta}{dt} = \omega \qquad \frac{d\varphi}{dt} = \omega_d$$

2. nonlinearity in the equations of motion

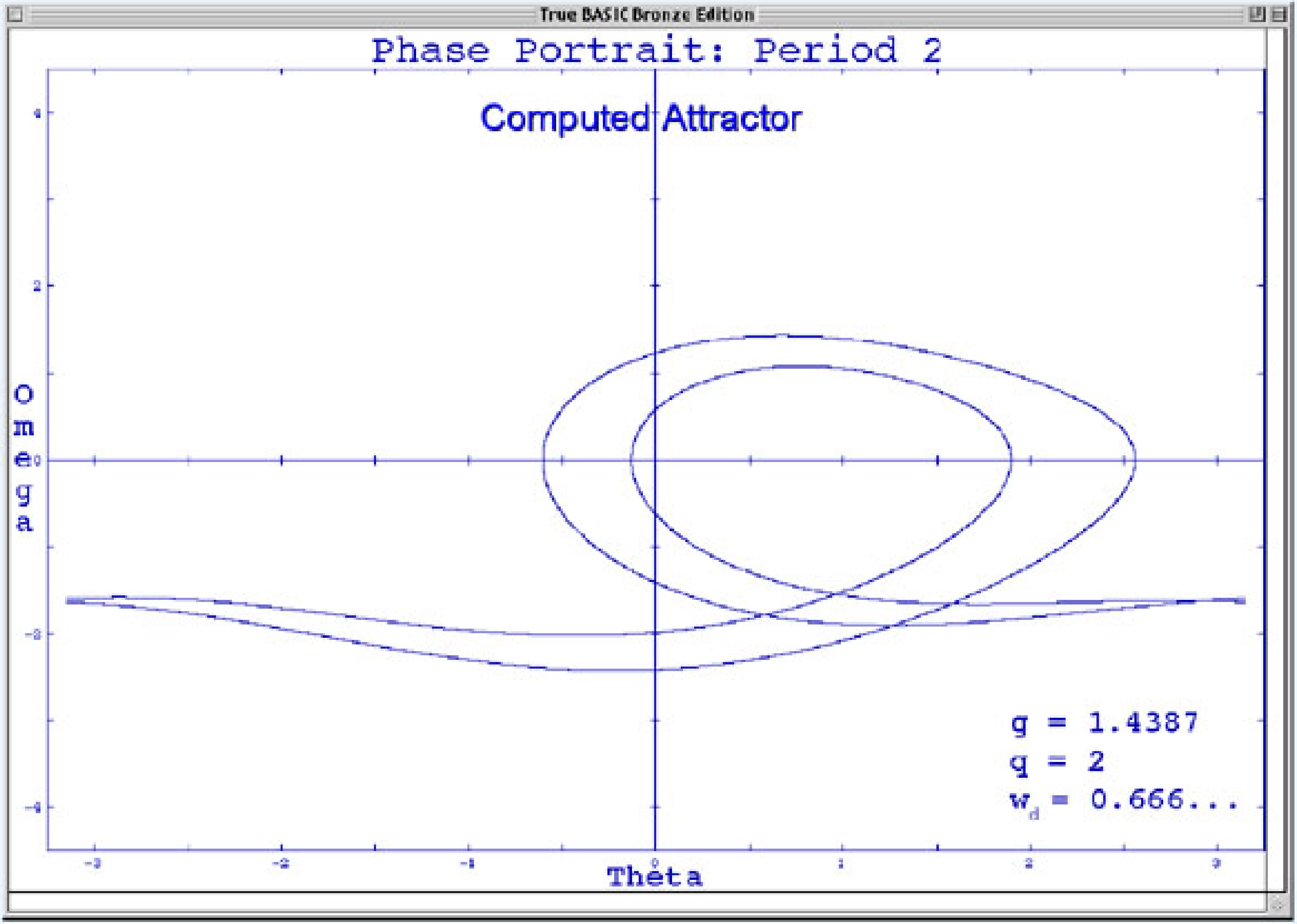
$$\sin\theta, \quad \sin(\omega_d t')$$

3. appropriate values for the parameters

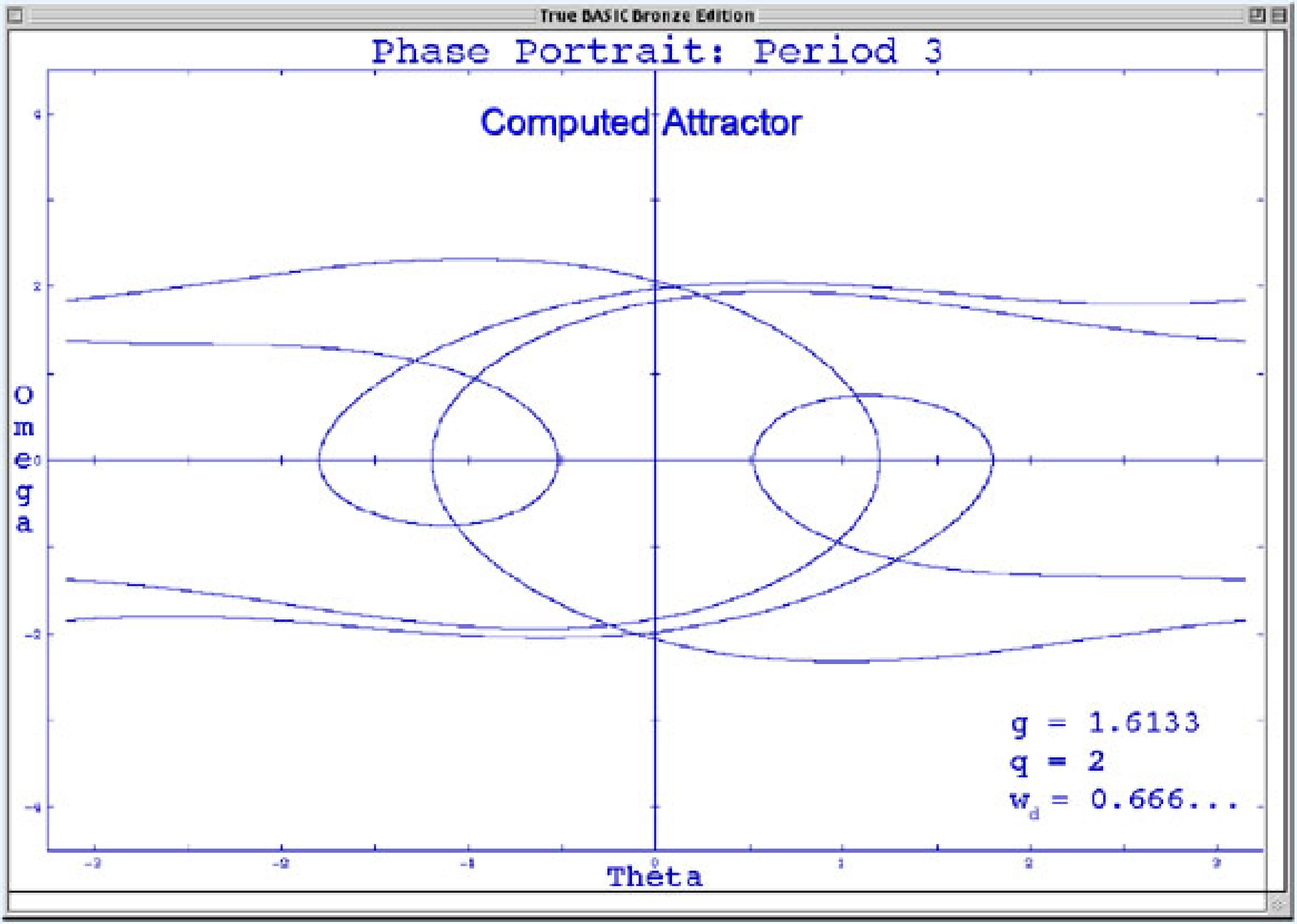
$$\frac{1}{q} = \left(\frac{1}{\omega_0}\right)\left(\frac{b}{I}\right) \qquad g = \frac{T}{\omega_0{}^2 I} = \frac{T}{T_{critical}} \qquad \omega_d = \frac{\omega_f}{\omega_0}$$



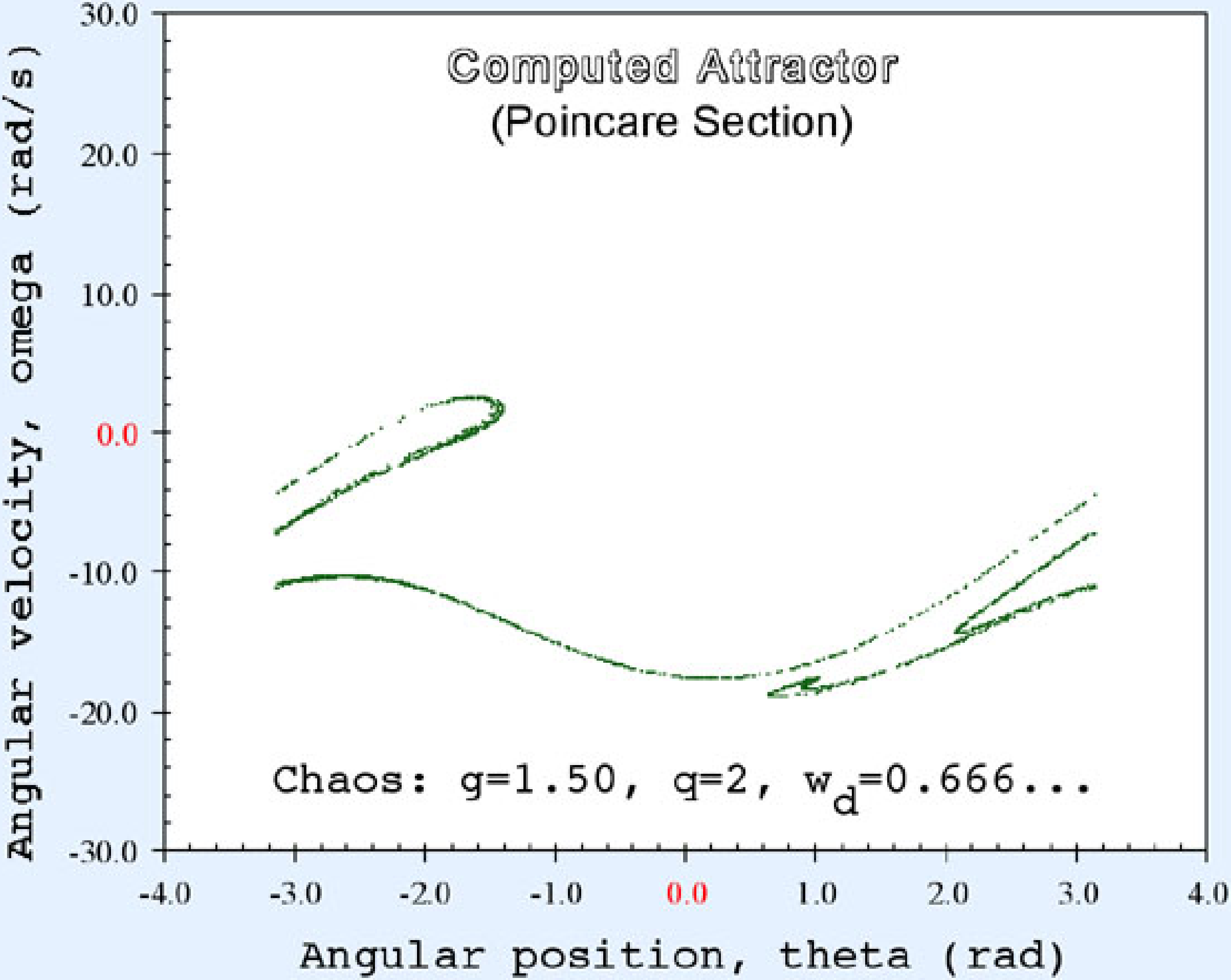
True BASIC Bronze Edition
Phase Portrait: Period 1
Computed Attractor
Omega
Theta
q = 0.9219
q = 2
w_d = 0.666...



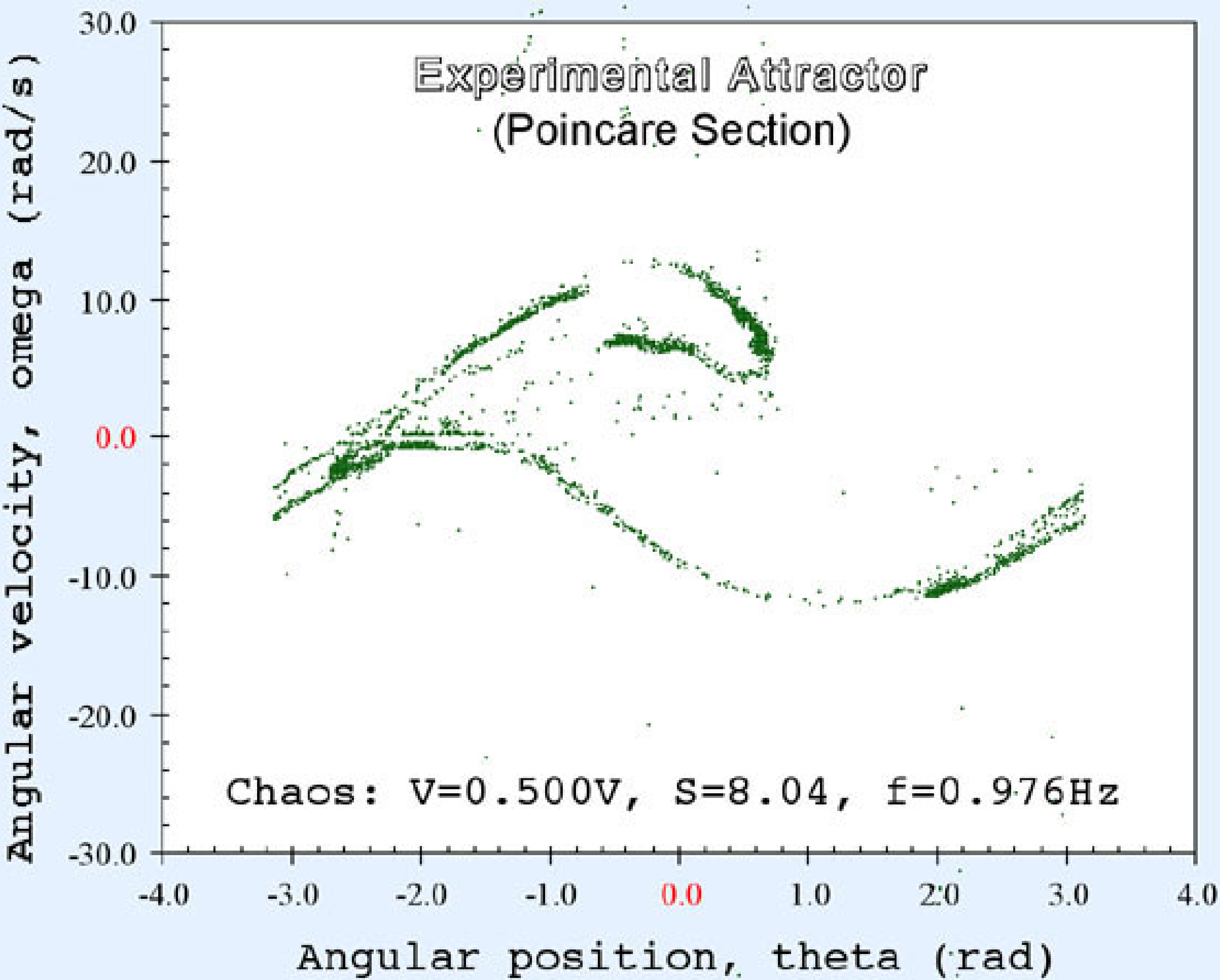
True BASIC Bronze Edition
Phase Portrait: Period 2
Computed Attractor
Omega
Theta
q = 1.4387
q = 2
w = 0.666...



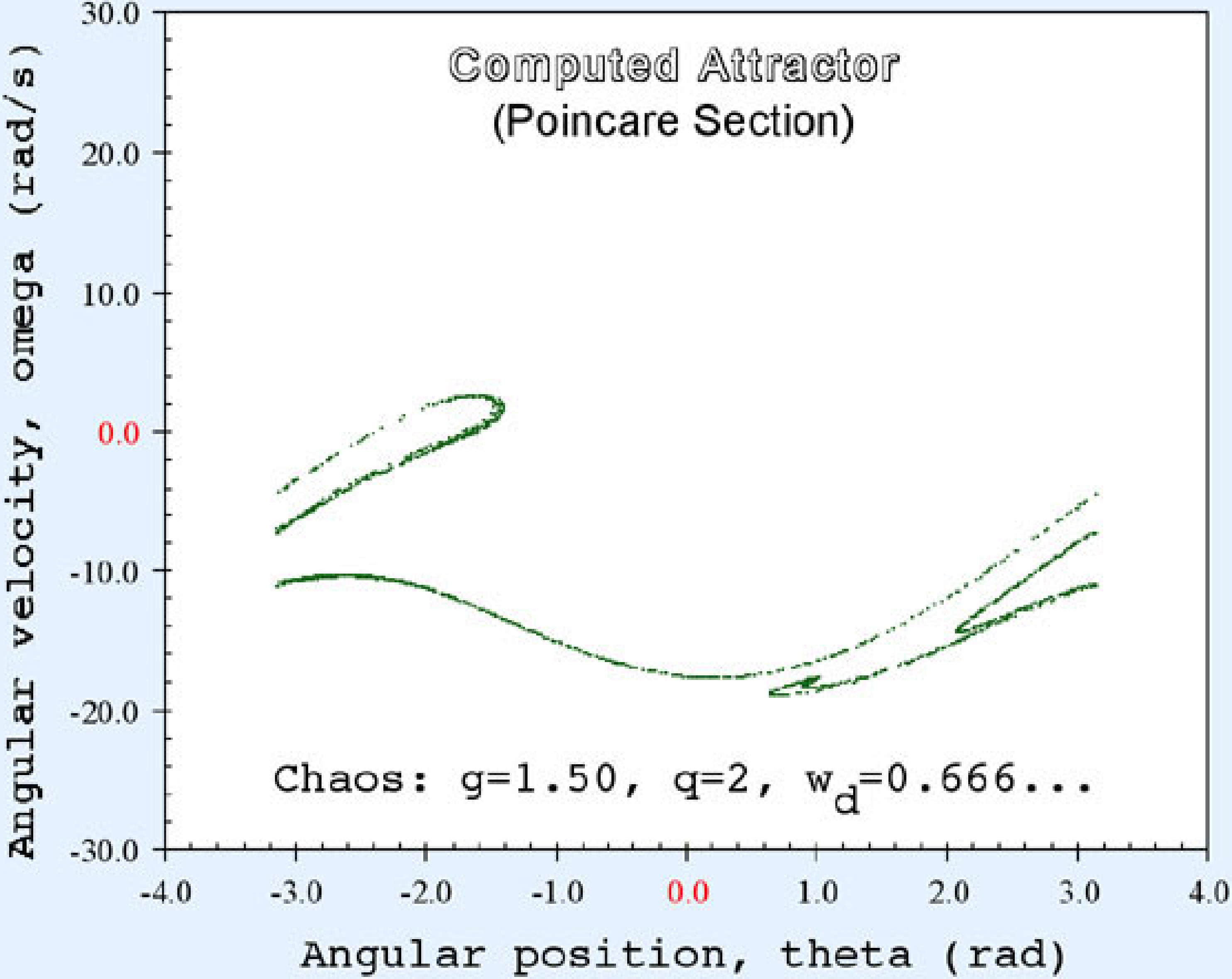
True BASIC Bronze Edition
Phase Portrait: Period 3
Computed Attractor
Omega
Theta
g = 1.6133
q = 2
w_d = 0.666...



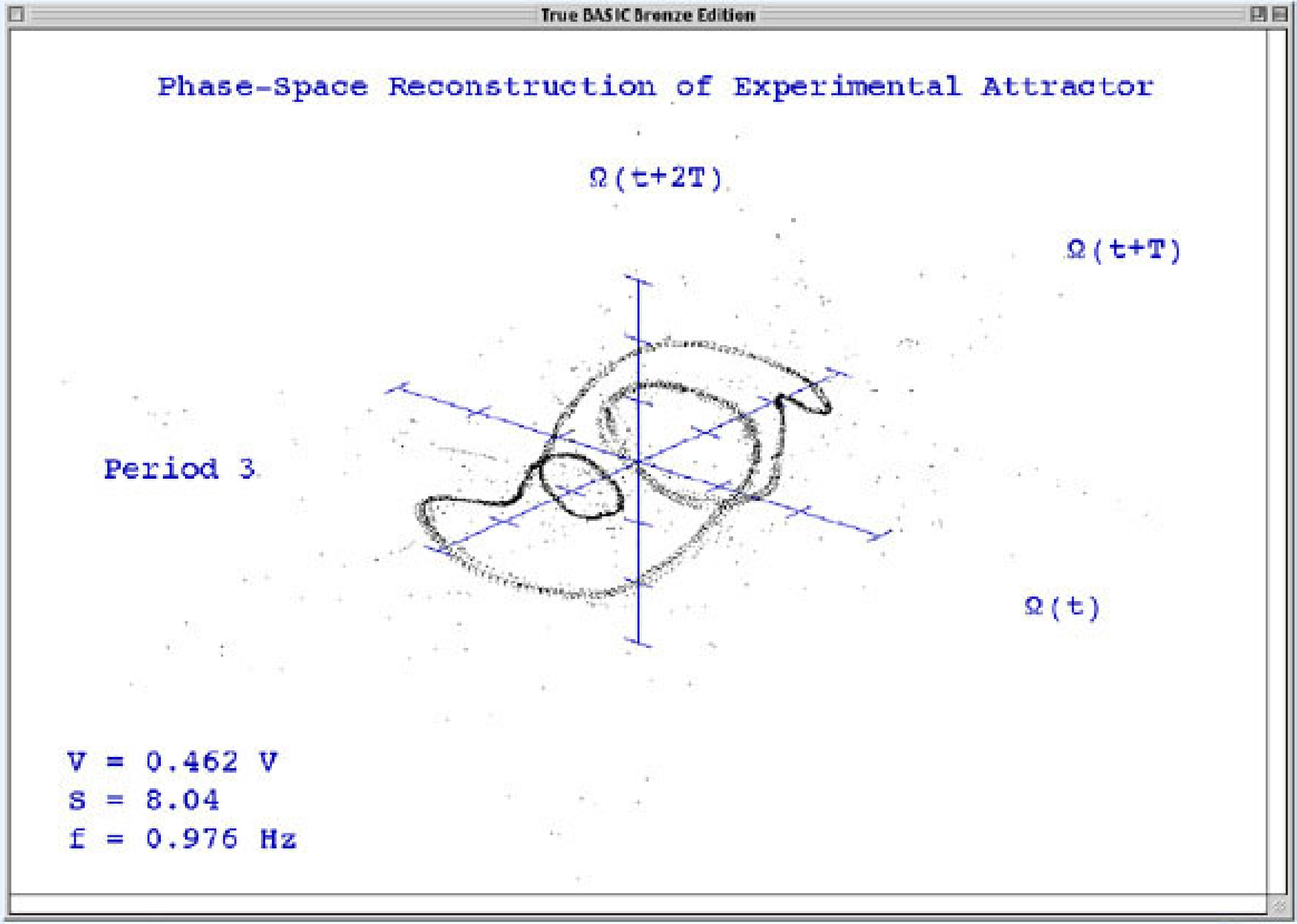
Computed Attractor
(Poincare Section)
Chaos: g=1.50, q=2, w$_d$=0.666...
Angular velocity, omega (rad/s)
Angular position, theta (rad)
30.0
20.0
10.0
0.0
-10.0
-20.0
-30.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0

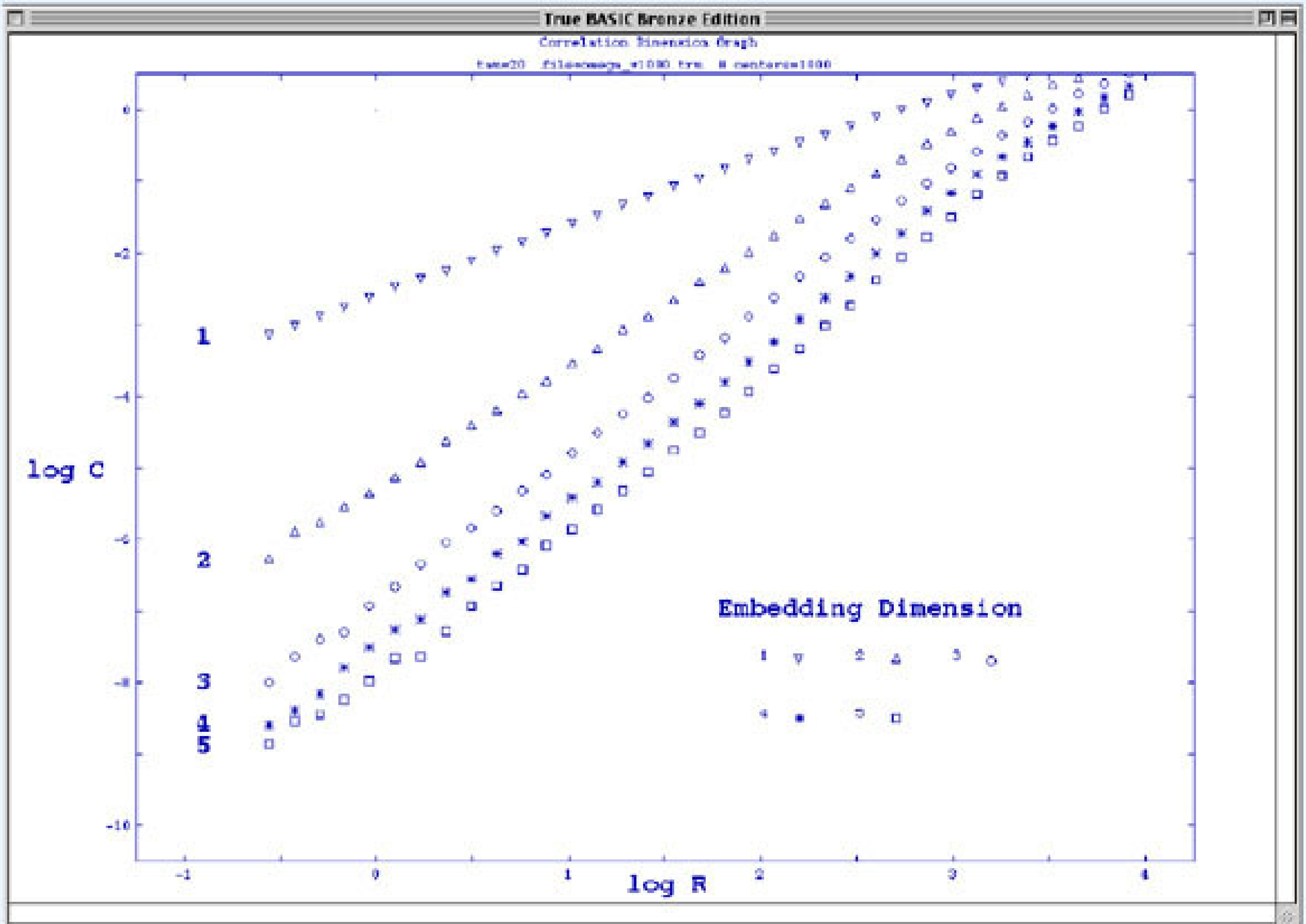


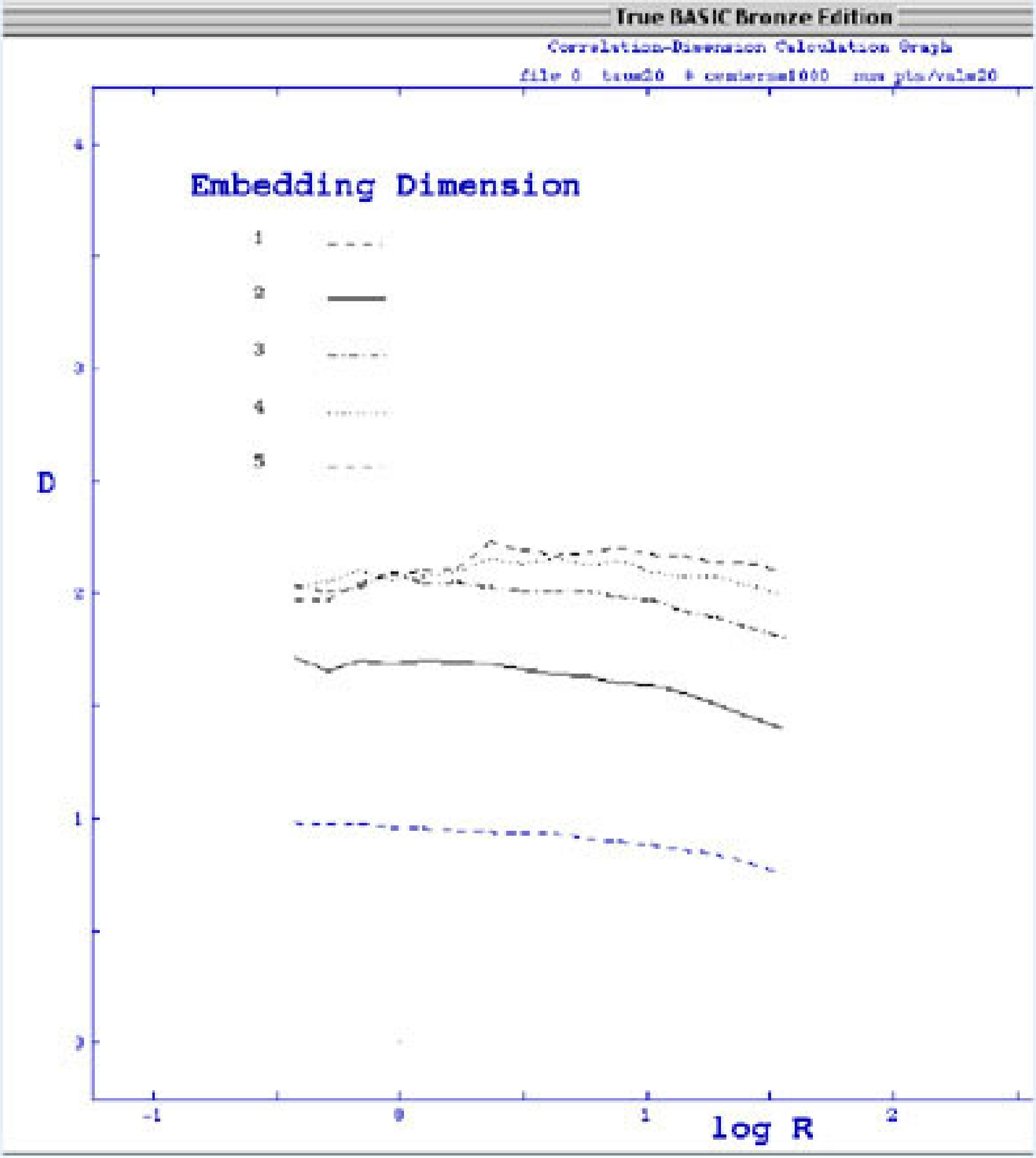
Experimental Attractor
(Poincare Section)
Chaos: V=0.500V, S=8.04, f=0.976Hz
Angular velocity, omega (rad/s)
Angular position, theta (rad)
30.0
20.0
10.0
0.0
-10.0
-20.0
-30.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0

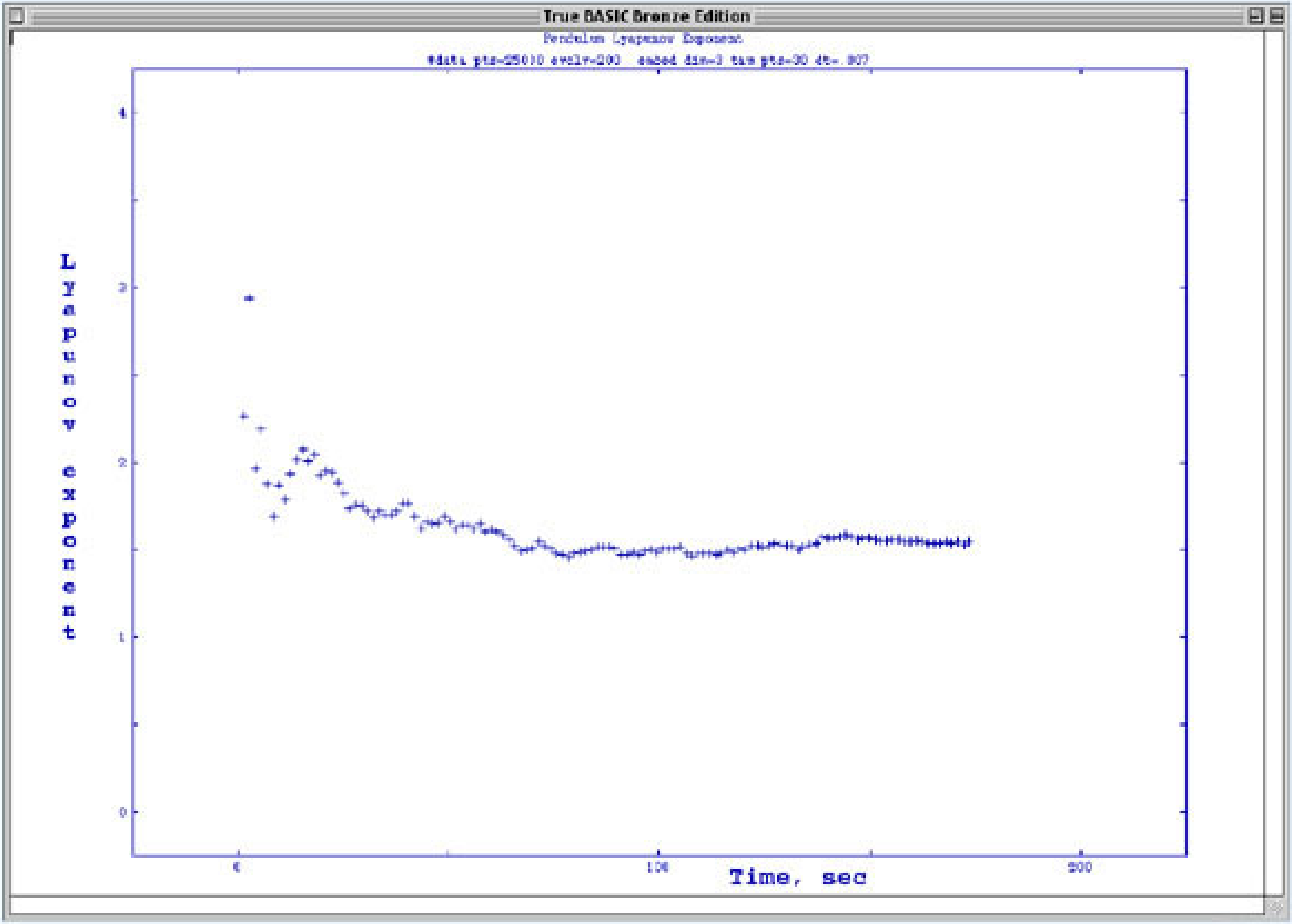


Computed Attractor
(Poincare Section)
Chaos: g=1.50, q=2, w_d=0.666...
Angular velocity, omega (rad/s)
Angular position, theta (rad)
30.0
20.0
10.0
0.0
-10.0
-20.0
-30.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
4.0

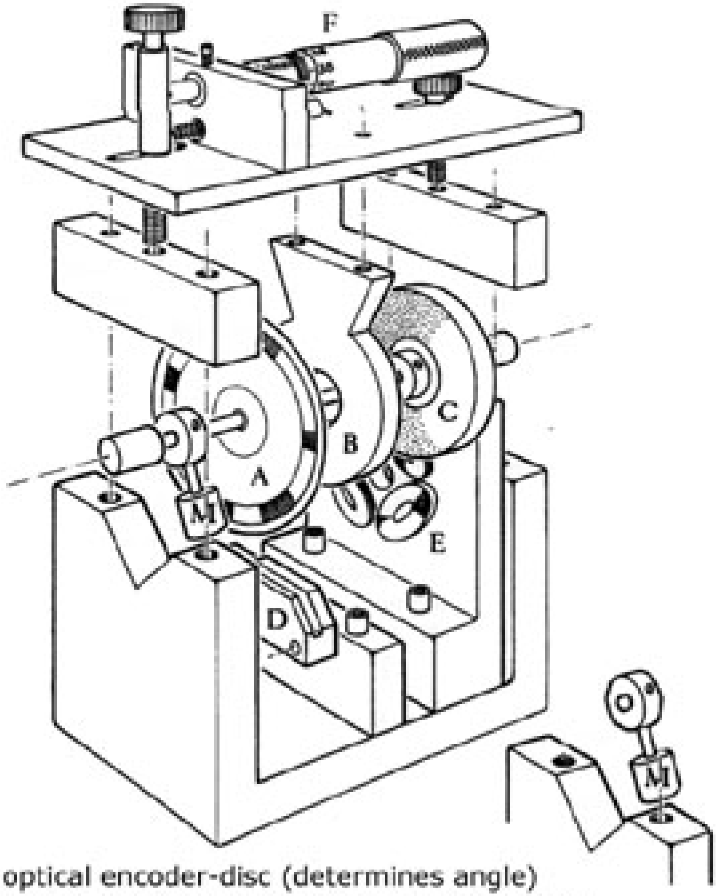

True BASIC Bronze Edition
Phase-Space Reconstruction of Experimental Attractor
Ω(t+2T)
Ω(t+T)
Period 3
Ω(t)
V = 0.462 V
S = 8.04
f = 0.976 Hz


True BASIC Bronze Edition
Correlation Dimension Graph
log C
log R
1
2
3
4
5
Embedding Dimension
1
2
3
4
5
0
-2
-4
-6
-8
-10
-1
0
1
2
3
4


True BASIC Bronze Edition
Correlation-Dimension Calculation Graph
Embedding Dimension
1
2
3
4
5
D
4
3
2
1
0
-1
0
1
2
log R


True BASIC Bronze Edition
Pendulum Lyapunov Exponent
Lyapunov exponent
Time, sec
0
1
2
3
4
0
100
200

"Exploded" View of the Chaotic Pendulum



A - optical encoder-disc (determines angle)
B - stationary copper plate (used for damping)
C - ring magnet (used for damping)
D - light emitter-detector assembly (reads angle)
E - two pairs of drive coils (parts of AC/DC motor)
F - micrometer (used to control damping)
M - pendulum mass (+ short rod = our "pendulum")

Chaos in a Pendulum

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Eugene Donev